Nilmanifolds: complex structures, geometry and deformations

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von

Sönke Rollenske
aus Bonn

1. Gutachter: Prof. Dr. Fabrizio Catanese
2. Gutachter: Prof. Dr. Jörg Winkelmann
3. Gutachter: Prof. Dr. Simon Salamon

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Abstract

We consider nilmanifolds with left-invariant complex structure and prove that in the generic case small deformations of such structures are again left-invariant.

The relation between nilmanifolds and iterated principal holomorphic torus bundles is clarified and we give criteria under which deformations in the large are again of such type. As an application we obtain a fairly complete picture in dimension three.

We show by example that the Frölicher spectral sequence of a nilmanifold may be arbitrarily non degenerate thereby answering a question mentioned in the book of Griffith and Harris.

On our way we prove Serre Duality for Lie algebra Dolbeault cohomology and classify complex structures on nilpotent Lie algebras with small commutator subalgebra.

MS Subject classification: 32G05; (32G08, 17B30, 53C30, 32C10)

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Dissertation mit dem Titel *Nilmanifolds: complex structures, geometry and deformations* selbstständig angefertigt habe. Alle benutzten Quellen und Hilfsmittel habe ich nach bestem Wissen und Gewissen kenntlich gemacht.

Dies ist mein erster Versuch, diese oder eine gleichartige Doktorprüfung abzulegen.

Sönke Rollenske — Bayreuth, den 31. Mai 2007

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Zusammenfassung

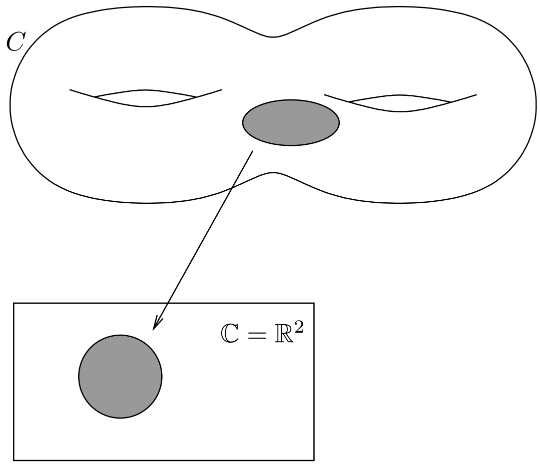
In dieser Arbeit werden wir eine spezielle Klasse von kompakten komplexen Mannigfaltigkeiten genauer studieren.

Seit Bernhard Riemann den Begriff der Mannigfaltigkeit in seiner Habilitationsschrift *Über die Hypothesen, welche der Geometrie zugrunde liegen* aus dem Jahr 1854 [Rie19] eingeführt hat, ist die Theorie der differenzierbaren und komplexen Mannigfaltigkeiten ein zentrales Thema der Mathematik.

Eine solche Mannigfaltigkeit ist ein (genügend guter) topologischer Raum, der aus offenen Teilmengen des $\mathbb{R}^n$ bzw. $\mathbb{C}^n$ mittels differenzierbarer bzw. holomorpher Abbildungen zusammengeklebt ist und somit eine natürliche Verallgemeinerung des $\mathbb{R}^n$ oder $\mathbb{C}^n$.

Da Mannigfaltigkeiten lokal aussehen wie der $\mathbb{R}^n$ bzw. $\mathbb{C}^n$ lassen sich die üblichen Methoden der Differential- und Integralrechnung auf Mannigfaltigkeiten übertragen; es treten jedoch qualitativ neue Phänomene auf, wenn man die globale Geometrie mit berücksichtigt.

Die Theorie der differenzierbaren Mannigfaltigkeiten beeinflusste und ermöglichte die Entwicklung der klassischen Mechanik und der Relativitätstheorie. Das Studium von Funktionen in einer komplexen Variablen führte nach der Entwicklung der analytischen Fortsetzung auf natürliche Weise zur Definition von eindimensionalen komplexen Mannigfaltigkeiten, die man heute zu Ehren Riemanns als Riemannsche Flächen bezeichnet.



Eine Riemannsche Fläche mit lokaler Karte.

Höherdimensionale komplexe Mannigfaltigkeiten traten implizit schon Anfang des 19. Jahrhunderts in den Arbeiten von Abel und Jacobi zu elliptischen Integralen auf: Eine Funktion f der Form

$$f(t) = \int_0^t \frac{1}{\sqrt{p(x)}} dx, \quad p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

lässt sich im allgemeinen nicht durch elementare Funktionen ausdrücken.

Betrachtet man jedoch die Riemannsche Fläche C in $\mathbb{P}^2_{\mathbb{C}}$ gegeben als Nullstellenmenge der Gleichung $y^2 = p(x)$ so erscheint die Funktion f als das Kurvenintegral

$$\int \frac{dx}{y}$$

auf C. Der natürliche Zielraum für solche Funktionen ist dann ein kompakter komplexer Torus, die Jacobische Varietät zu C, deren präzise Definition auf Lefschetz in den zwanziger Jahren des letzten Jahrhunderts zurückgeht.

Die Entwicklung der modernen Theorie der komplexen und differenzierbaren Mannigfaltigkeiten setzte dann Anfang des 20. Jahrhunderts ein, getragen von der Entwicklung der Topologie und den fortschreitenden Erkenntnissen in der komplexen Analysis in mehreren Veränderlichen, der Differentialtopologie und Differentialgeometrie.

Weil wir $\mathbb{C}^n$ mit $\mathbb{R}^{2n}$ identifizieren können und jede holomorphe Funktion insbesondere differenzierbar ist, können wir jede n-dimensionale komplexe Mannigfaltigkeit X auch als $2n$-dimensionale differenzierbare Mannigfaltigkeit M betrachten. In jedem Punkt $x \in M$ ist der Tangentialraum T_xM ein komplexer Vektorraum und die Multiplikation mit i liefert einen Endomorphismus des Tangentialraumes

$$J : T_xM \to T_xM, \qquad J\xi = i\xi.$$

Fasst man die Tangentialräume in den einzelnen Punkten zum Tangentialbündel $TM = \dot{\bigcup}_{x\in M} T_xM$ zusammen, so erhalten wir einen globalen Endomorphismus

$$J : TM \to TM, \qquad J^2 = -id_{TM},$$

der in jedem Punkt durch die Multiplikation mit i wirkt.

Ein solcher Endomorphismus J mit $J^2 = -id_{TM}$ auf einer differenzierbaren Mannigfaltigkeit M gerader Dimension heißt fast komplexe Struktur.

Nach einem Resultat von Newlander und Nierenberg induziert eine fast komplexe Struktur J genau dann eine (in diesem Fall eindeutig bestimmte) Struktur einer komplexen Mannigfaltigkeit auf M, falls sie integrabel ist, d.h. für alle Vektorfelder X, Y auf M gilt

$$[JX, JY] = [X, Y] + J[JX, y] + J[X, JY]$$

wobei $[\cdot,\cdot]$ die Lieklammer ist.

Diese Integrabilitätsbedingung wird in unseren Untersuchungen eine große Rolle spielen, denn sie erlaubt uns, die Existenz einer globalen komplexen Struktur auf M anhand einer lokalen Bedingung zu testen, die sich mit Methoden der linearen Algebra studieren lässt.

Schon beim Studium der Riemannschen Flächen wurde bemerkt, dass zwei verschiedene komplexe Mannigfaltigkeiten die gleiche zugrundeliegende differenzierbare Mannigfaltigkeit haben können oder, mit anderen Worten, auf einer differenzierbaren Mannigfaltigkeit M kann es essentiell unterschiedliche integrable fast komplexe Strukturen geben.

Eine sehr allgemeine Frage in der Theorie der komplexen Mannigfaltigkeiten ist nun die folgende: Sei M eine kompakte, differenzierbare Mannigfaltigkeit und sei

$$\mathcal{C} := \{J \in \mathrm{End}(TM) \mid J^2 = -id_{TM}, J \text{ integrable fast komplexe Struktur}\}$$

der Raum aller komplexen Strukturen auf M; was können wir über $\mathcal{C}$ und seine Elemente sagen, falls $\mathcal{C}$ nicht leer ist?

Falls M reelle Dimension 2 hat und orientierbar ist, so sind der Raum der komplexen Strukturen $\mathcal{C}$ und verfeinerte Versionen davon detailiert studiert und beschrieben worden; unter anderen hat $\mathcal{C}$ genau zwei Komponenten, die den beiden möglichen Orientierungen entsprechen. Ein geeigneter Quotient, der die integrablen fast komplexen Strukturen effektiv parametrisiert, ist biholomorph zu einer offenen Teilmenge des $\mathbb{C}^n$ für ein geeignetes n, das nur von der topologischen Gestalt von M abhängt.

Im allgemeinen ist die Situation wesentlich komplizierter.

Um die Fragestellungen einzugrenzen, können wir mit einer kompakten, komplexen Mannigfaltigkeit $X = (M, J)$, betrachtet als differenzierbare Mannigfaltigkeit M mit integrabler fast komplexer Struktur J, beginnen und versuchen, alle komplexen Strukturen auf M in einer kleinen Umgebung von J (kleine Deformationen) beziehungsweise in der gleichen Zusammenhangskomponente wie J (Deformationen im Großen) zu verstehen.

Vom geometrischen Standpunkt aus sagen wir: Zwei kompakte, komplexe Mannigfaltigkeiten X und X' heißen direkt deformationsäquivalent $X \sim_{\mathrm{def}} X'$, wenn es eine eigentliche, flache Familie $\pi : \mathcal{M} \to \mathcal{B}$ von kompakten, komplexen Mannigfaltigkeiten über einem irreduziblen komplexanalytischen Raum $\mathcal{B}$ gibt, so dass $X \cong \pi^{-1}(b)$ und $X' \cong \pi^{-1}(b')$ für zwei Punkte $b, b' \in \mathcal{B}$. Nach einem Satz von Ehresmann sind alle Fasern von π diffeomorph und betrachtet man den Fall $\mathcal{B} = \Delta = \{z \in \mathbb{C} \mid |z| < 1\}$, so hat man eine Familie von komplexen Strukturen, die von einem Parameter $t \in \Delta$ abhängt.

Die Mannigfaltigkeit X' heißt Deformation im Großen von X, falls beide in der gleichen Äquivalenzklasse bezüglich der von $\sim_{\mathrm{def}}$ erzeugten Äquivalenzrelation sind. Dies ist genau dann der Fall, wenn X und X' in der gleichen Zusammenhangskomponente von $\mathcal{C}$ liegen.

Die Deformationen einer gegebenen Mannigfaltigkeit zu studieren, ist im Allgemeinen sehr schwer; während es aber eine universelle Methode, entwickelt von Kuranishi, Kodaira und Spencer [KS58, Kur62], zur Bestimmung aller kleinen Deformationen gibt, fehlt ein solcher allgemeiner Ansatz für Deformationen im Großen.

Sogar die scheinbar selbstverständliche Tatsache, dass jede Deformation im Großen eines komplexen Torus wieder ein solcher ist, wurde erst 2002 von

Catanese [Cat02] vollständig bewiesen. Der Ausgangspunkt für diese Dissertation waren seine darauf folgenden Untersuchungen zu Deformationen im Großen von Torus-Prinzipalbündeln in [Cat04].

Es stellt sich heraus [CF06], dass der richtige Kontext zur Verallgemeinerung dieser Ergebnisse die Theorie der linksinvarianten komplexen Strukturen auf Nilmannigfaltigkeiten, d.h. kompakten Quotienten reeller nilpotenter Liegruppen, ist.

Nilmannigfaltigkeiten mit linksinvarianter komplexer Struktur sind eine wichtige Quelle von Beispielen in der komplexen Differentialgeometrie. In diese Klasse fallen unter anderem die Kodaira-Thurston-Mannigfaltigkeiten, die die ersten Beispiele von Mannigfaltigkeiten sind, die sowohl eine komplexe als auch eine symplektische Struktur, jedoch keine Kählerstruktur zulassen. Genauer gilt, dass eine Nilmannigfaltigkeit M genau dann mit einer Kählerstruktur versehen werden kann, wenn M ein Torus ist [BG88, Has89].

Die Anzahl der Schritte, nach denen die Frölicher-Spektralsequenz, die Dolbeault-Kohomologie mit deRham-Kohomologie verbindet, degeneriert, ist ein Maß dafür, wie weit eine Mannigfaltigkeit davon entfernt ist, eine Kählermannigfaltigkeit zu sein. In Abschnitt 1.4 werden wir anhand eines Beispiels zeigen, dass Nilmannigfaltigkeiten in diesem Sinne beliebig weit entfernt von Kählermannigfaltigkeiten sein können.

Obwohl sich jedes iterierte holomorphe Torus-Prinzipalbündel als Nilmannigfaltigkeit mit linksinvarianter komplexer Struktur schreiben lässt, ist das umgekehrt nicht möglich. Es stellt sich sogar heraus, dass eine kleine Deformation eines iterierten Prinzipalbündels nicht notwendig ein solches ist (Beispiel 1.14).

Wir müssen uns also mit den folgenden drei Fragen befassen:

- Welches sind die kleinen Deformationen einer Nilmannigfaltigkeit mit linksinvarianter komplexer Struktur?

- Wann induziert eine linksinvariante komplexe Struktur auf einer Nilmannigfaltigkeit eine Struktur als iteriertes holomorphes Torus-Prinzipalbündel?

- Unter welchen Bedingungen kann man die Deformationen im Großen eines iterierten holomorphen Torus-Prinzipalbündels kontrollieren?

Die fundamentalen Ergebnisse der Theorie der linksinvarianten komplexen Strukturen auf Nilmannigfaltigkeiten finden sich vor allem in den im Literaturverzeichnis zitierten Arbeiten von Console, Cordero, Fernandez, Fino, Grantcharov, Gray, McLaughlin, Pedersen, Poon, Salamon und Ugarte.

In Abschnitt 1 werden wir die bekannten Resultate zusammenfassen, wobei wir den Aspekt der komplex-geometrischen Struktur besonders betonen. Wir werden Nilmannigfaltigkeiten durch ein Tripel $M = (\mathfrak{g}, J, \Gamma)$ beschreiben, wobei $\mathfrak{g}$ die Liealgebra zu einer einfach zusammenhängenden, nilpotenten Liegruppe G, Γ ein Gitter in G und J eine integrable komplexe Struktur auf $\mathfrak{g}$ sind (vergleiche Abschnitt 1.1).

Die zu Grunde liegende Philosophie ist, dass die Geometrie und die Deformationen der kompakten, differenzierbaren Mannigfaltigkeit $\Gamma\backslash G$, versehen mit der von J induzierten komplexen Struktur, vollständig vom Zusammenspiel von $\mathfrak{g}$, J und der von $\log\Gamma \subset \mathfrak{g}$ erzeugten $\mathbb{Q}$-Unteralgebra bestimmt sein sollten.

Um die kleinen Deformationen einer Nilmannigfaltigkeit zu bestimmen, brauchen wir eine gute Kontrolle über die Dolbeault-Kohomologie des holomorphen Tangentialbündels.

In Abschnitt 2 entwickeln wir eine Liealgebra-Dolbeault-Kohomologie mit Werten in integrablen Moduln (Definition 2.1) und beweisen Serre-Dualität in diesem Kontext (Theorem 2.18). Da es bekannt ist, dass die gewöhnliche Dolbeault-Kohomologie sich mittels linksinvarianter Differentialformen berechnen lässt, falls die komplexe Struktur J generisch ist [CF01, CFGU00], können wir in diesen Fällen die Kohomologie des Tangentialbündels mit der des Komplexes

$$0 \to \mathfrak{g}^{1,0} \xrightarrow{\bar\partial} \mathfrak{g}^{*0,1} \otimes \mathfrak{g}^{1,0} \xrightarrow{\bar\partial} \Lambda^2\mathfrak{g}^{*0,1} \otimes \mathfrak{g}^{1,0} \xrightarrow{\bar\partial} \dots$$

identifizieren.

Die Theorie von Kuranishi liefert dann eine sehr explizite Beschreibung der kleinen Deformationen:

Theorem 3.5 — *Sei $M_J = (\mathfrak{g}, J, \Gamma)$ eine Nilmannigfaltigkeit mit linksinvarianter komplexer Struktur, so dass $\iota : H^{1,q}((\mathfrak{g}, J), \mathbb{C}) \to H^{1,q}(M_J)$ für alle q ein Isomorphismus ist. Dann sind alle kleinen Deformationen wieder Nilmannigfaltigkeiten mit linksinvarianter fast komplexer Struktur der Form $M_{J'} = (\mathfrak{g}, J', \Gamma)$.*

Dies verallgemeinert das analoge Resultat von Console, Fino und Poon [CFP06] für abelsche komplexe Strukturen (Siehe auch [MPPS06]).

Die reelle Geometrie von Nilmannigfaltigkeiten ist gut bekannt, aber, um eine ausreichende Kontrolle über die komplexe Geometrie von (M, J) zu bekommen, brauchen wir die Existenz einer stabilen Torusbündel-Reihe (Definition 1.10) in der zugehörigen Liealgebra. Geometrisch lässt sich dieser Begriff wie folgt interpretieren:

Jede reelle nilpotente Liegruppe G lässt sich durch Untergruppen, z.B. die aufsteigende zentrale Reihe, filtrieren:

$$G \supset H_s \supset \cdots \supset H_1 \supset H_0 = \{1\},$$

so dass H_k/H_{k-1} im Zentrum von G/H_{k-1} enthalten ist, und $\Gamma \cap H_k$ ein Gitter in H_k unabhängig von der Wahl des Gitters $\Gamma \subset G$ ist.

Mit anderen Worten, jede kompakte reelle Nilmannigfaltigkeit kann durch einen Turm von differenzierbaren Torus-Prinzipalbündeln $\pi_k : M_k \to M_{k+1}$ beschrieben werden:

$$\begin{array}{ccc}
T_1 & \hookrightarrow & M_1 = M \\
 & & \downarrow \pi_1 \\
T_2 & \hookrightarrow & M_2 \\
 & & \downarrow \pi_2 \\
 & & \vdots \\
 & & \downarrow \\
T_{s-1} & \hookrightarrow & M_{s-1} \\
 & & \downarrow \pi_{s-1} \\
 & & M_s = T_s
\end{array}$$

Hierbei ist der kompakte Torus T_k der Quotient von H_k/H_{k-1} modulo des Gitters und M_k ist die kompakte Nilmannigfaltigkeit, die man als Quotient von G/H_{k-1} modulo des Bildes von Γ bekommt.

Solch eine Zerlegung in ein iteriertes Prinzipalbündel ist in keiner Weise eindeutig und wir sagen, dass $\mathfrak{g}$ eine stabile Torusbündel-Reihe hat, falls die Untergruppen H_k so gewählt werden können, dass jede linksinvariante komplexe Struktur auf M ebensolche auf allen M_k induziert bezüglich derer die Abbildungen $\pi_k : M_k \to M_{k+1}$ holomorph sind.

Dieses Phänomen ist nicht so speziell wie es auf den ersten Blick scheinen mag, zum Beispiel hat jedes nicht triviale holomorphe Torus-Prinzipalbündel über einer elliptischen Kurve diese Eigenschaft. In Abschnitt 5 werden wir die unter der Bedingung $\dim[\mathfrak{g}, \mathfrak{g}] \leq 3$ auftretenden Fälle klassifizieren.

Unser Zugang gibt einen neuen Beweis eines großen Teils der Klassifikation komplexer Strukturen auf 6-dimensionalen reellen nilpotenten Liealgebren nach Salamon und Ugarte [Sal01, Uga04].

Es stellt sich heraus, dass die Existenz einer stabilen Torusbündel-Reihe ein guter Ausgangspunkt ist, um Deformationen im Großen zu studieren. In diesem Fall finden wir nämlich die Faserung $\pi : M \to T_s$ als (topologisch) festen Quotienten der Albanesevarietät wieder und wir werden in Abschnitt 4 alle Deformationen im Großen bestimmen, wenn die Fasern von π genügend gute Eigenschaften haben.

Theorem 4.12 — *Sei G eine einfach zusammenhängende, reelle, nilpotente Liegruppe mit Liealgebra $\mathfrak{g}$ und sei $\Gamma \subset G$ ein Gitter. Weiterhin gelte:*

(*i*) *$\mathfrak{g}$ besitzt eine stabile Torusbündel-Reihe $(\mathcal{S}^i\mathfrak{g})_{i=0,\dots,t}$ (siehe Definition 1.10).*

(*ii*) *Die Nilmannigfaltigkeiten vom Typ* $(\mathcal{S}^{t-1}\mathfrak{g}, J, \Gamma\cap\exp(\mathcal{S}^{t-1}\mathfrak{g}))$ *sind eine gute Faserklasse (siehe Definition 4.9).*

Dann ist jede Deformation im Großen M' *einer Nilmannigfaltigkeit* $M = (\mathfrak{g}, J, \Gamma)$ *wieder eine solche, d.h.* $M' = (\mathfrak{g}, J', \Gamma)$*.*

Um die Methoden in [Cat04] zu verallgemeinern müssen wir das Problem überwinden, dass die Dimension der Albanesevarietät sich in einer Familie von Mannigfaltigkeiten ändern kann.

In Abschnitt 6 werden wir unsere Ergebnisse über Deformationen mit der Klassifikation in Abschnitt 5 kombinieren und dadurch viele nicht triviale Beispiele angeben können.

Wir bezeichnen mit $\mathcal{Z}\mathfrak{g}$ das Zentrum und mit $\mathcal{C}^i\mathfrak{g}$ die absteigende Zentralreihe in einer Liealgebra $\mathfrak{g}$.

Theorem 6.1 — *Sei* $M = (\mathfrak{g}, J, \Gamma)$ *eine Nilmannigfaltigkeit mit linksinvarianter komplexer Struktur.*

(*i*) *Jede kleine Deformation von* M *ist wieder eine Nilmannigfaltigkeit mit linksinvarianter komplexer Struktur von der Form* $M' = (\mathfrak{g}, J', \Gamma)$*, falls eine der folgenden Bedingungen gilt:*

- $\dim \mathcal{C}^1\mathfrak{g} \leq 2$.
- $\dim \mathcal{C}^1\mathfrak{g} = 3$ *und* $\mathfrak{g}$ *ist 4-Schritt nilpotent.*
- $\dim \mathcal{C}^1\mathfrak{g} = 3$*,* $\mathfrak{g}$ *ist 3-Schritt nilpotent und* $\dim(\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}) = 2$*.*

(*ii*) *Jede Deformation im Großen von* M *ist wieder eine Nilmannigfaltigkeit mit linksinvarianter komplexer Struktur von der Form*

$$M' = (\mathfrak{g}, J', \Gamma),$$

falls eine der folgenden Bedingungen erfüllt ist:

- $\mathfrak{g}$ *ist abelsch.*
- $\dim \mathcal{C}^1\mathfrak{g} \leq 1$*.*
- $\mathfrak{g}$ *erfüllt die Bedingungen von Proposition 5.4.*
- $\dim \mathcal{C}^1\mathfrak{g} = 2$*,* $\mathfrak{g}$ *ist 2-Schritt nilpotent und* $\dim(\mathcal{Z}\mathfrak{g})$ *ungerade oder gleich 2.*
- $\mathfrak{g}$ *ist 2-Schritt nilpotent und hat eine stabile Torusbündelreihe*

$$\mathfrak{g} \supset \mathcal{S}^1\mathfrak{g} \supset 0.$$

- $\dim \mathcal{C}^1\mathfrak{g} = 2$ *und* $\mathfrak{g}$ *ist 3-Schritt nilpotent.*
- $\dim \mathcal{C}^1\mathfrak{g} = 3$*,* $\mathfrak{g}$ *ist 3-Schritt nilpotent und weiterhin*
  - $\dim \mathcal{Z}^1\mathfrak{g}$ *ist ungerade oder gleich 2.*

– $\dim \mathcal{Z}^2\mathfrak{g}$ *ist gerade.*
– $\dim(\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}) = 2$.
– $\dim \mathcal{C}^1(\mathcal{Z}^2\mathfrak{g}) = 1$.

Die letzte Bedingung ist automatisch erfüllt, falls $\dim \mathcal{C}^2\mathfrak{g} = 1$.

- $\mathfrak{g}$ *ist 3-Schritt nilpotent und hat eine stabile Torusbündel-Reihe der Form*

$$\mathfrak{g} \supset \mathcal{S}^2\mathfrak{g} \supset \mathcal{S}^1\mathfrak{g} \supset 0$$

so dass $\dim(\mathcal{C}^1(\mathcal{S}^2\mathfrak{g})) = 1$.

Des Weiteren haben M und M' jeweils eine Struktur als iteriertes holomorphes Torus-Prinzipalbündel, dessen Faserdimensionen nicht von der komplexen Struktur abhängen, falls eine der Bedingungen in (ii) erfüllt ist.

Zusammen mit der Klassifikation komplexer Strukturen auf 6-dimensionalen, reellen, nilpotenten Liealgebren [Sal01, Uga04] erhalten wir (in der Notation von Abschnitt 6.2):

Theorem 6.3 — *Sei $M = (\mathfrak{g}, J, \Gamma)$ eine komplex 3-dimensionale Nilmannigfaltigkeit mit linksinvarianter komplexer Struktur. Wenn $\mathfrak{g}$ nicht aus der Menge $\{\mathfrak{h}_7, \mathfrak{h}_{19}^-, \mathfrak{h}_{26}^+\}$ ist, dann ist M ein iteriertes holomorphes Torus-Prinzipalbündel aus der folgenden Tabelle:*

Basis	*Faser*	*zugehörige Liealgebren*
3-Torus	-	$\mathfrak{h}_1$
2-Torus	*elliptische Kurve*	$\mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6$
elliptische Kurve	*2-Torus*	$\mathfrak{h}_8$
Kodairafläche	*elliptische Kurve*	$\mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_{13}, \mathfrak{h}_{14}, \mathfrak{h}_{15}, \mathfrak{h}_{16}$

Insbesondere ist die Geometrie von M schon alleine von der Liealgebra bestimmt. Jede Deformation im Großen von M hat die gleiche Struktur.

Falls $\mathfrak{g} = \mathfrak{h}_7$ so gibt es eine dichte Teilmenge des Raumes der linksinvarianten komplexen Strukturen, so dass M ein Prinzipalbündel von elliptischen Kurven über einer Kodairafläche für die entsprechenden komplexen Strukturen ist; aber es gibt komplexe Strukturen, für welche M kein holomorphes Prinzipalbündel ist.

In den verbleibenden Fällen $\mathfrak{h}_{19}^-$ und $\mathfrak{h}_{26}^+$ gibt es nie eine holomorphe Torus-Prinzipalbündel-Struktur.

Um immer vor Augen zu haben, welche Bedeutung die Resultate der einzelnen Abschnitte für das angestrebte Ergebnis haben, werden wir diese jeweils direkt auf holomorphe Torus-Prinzipalbündel über elliptischen Kurven anwenden.

Wir werden die zu Grunde liegende Liealgebra sowie die stabile Torusbündel-Reihe in Beispiel 1.13 beschreiben und die kleinen Deformationen in Korollar 3.6 bestimmen. Der Beschreibung der Albanesevarietät ist Beispiel 4.4 gewidmet, und in 4.14 werden wir zeigen, dass jede Deformation im Großen vom selben Typ ist.

Aber selbst wenn wir nur die Anwendungen auf Nilmannigfaltigkeiten im Sinn haben, so sind unsere Resultate doch allgemeiner: Die Resultate über Albanesequotienten lassen sich auch auf andere holomorphe Bündel über Tori anwenden und die Liealgebra-Dolbeault-Kohomologie könnte sich im Studium der graduierten Differentialalgebren, die aus nilpotenten Liealgebren mit komplexer Struktur konstruiert werden, als nützlich erweisen (vgl. [Poo04, And06]).

Es wäre auch interessant, den Raum der komplexen Strukturen genauer zu beschreiben und z.B. zu untersuchen, ob er glatt und vollständig ist. Auch sollte die Konstruktion eines Modulraums unter gewissen Bedingungen möglich sein. Dies ist in einigen Fällen von verschiedenen Autoren gemacht worden [Cat04, CF06, KS04, GMPP04].

Danksagung

Zuallererst möchte ich Fabrizio Catanese für die Einführung in dieses Forschungsgebiet, die andauernde Unterstützung und viele wertvolle Hinweise danken. Simon Salamon hat eine verläufige Version dieser Arbeit kritisch gelesen, wertvolle Literaturhinweise gegeben und geholfen, an mehreren Stellen die Präsentation zu verbessern. Thomas Peternell hat bemerkt, dass Abschnitt 1.4 eine in [GH78] angesprochene Frage beantwortet. Jörg Winkelmann wies auf eine Vereinfachung in der Definition der Albanesevarietät hin.

Ein unschätzbarer Literaturhinweis von Oliver Goertsches eröffnete eine neue Sichtweise auf das Thema. Mein Dank gilt weiterhin Ingrid Bauer-Catanese, Andreas Höring, Christian Liedtke, Stefania Lo Forte, Michael Lönne und Eva Nowak sowie meiner ganzen Familie. Ohne die Unterstützung von Christina Thelen wäre diese Arbeit nie geschrieben worden.

Einen Teil der letzten Korrekturen konnte ich während eines Aufenthaltes am Max Planck Institut in Bonn erledigen.

Introduction

A very general question in the theory of complex manifolds is the following: let M be a compact, differentiable manifold and let

$$\mathcal{C} := \{J \in \mathrm{End}(TM) \mid J^2 = -id_{TM}, J \text{ a complex structure}\}$$

be the space of all complex structures on M. What can we say about $\mathcal{C}$ and its connected components?

If $\mathcal{C}$ is nonempty consider a compact, complex manifold $X = (M, J)$ where M is the underlying differentiable manifold and J is an integrable almost complex structure on M. The theory developed by Kodaira and Spencer in the 50's [KS58] and culminating in the theorem of Kuranishi [Kur62] suceeds in giving a rather precise description of a slice of $\mathcal{C}$ containing J which is transversal to the orbit of the natural action of $\mathrm{Diff}^+(M)$, called the Kuranishi slice.

While we have this powerful tool for the study of small deformations there is no general method available to study the connected compontents of $\mathcal{C}$.

From another point of view we say that two compact, complex manifolds X and X' are directly deformation equivalent $X \sim_{\mathrm{def}} X'$ if there exists an irreducible, flat family $\pi : \mathcal{M} \to \mathcal{B}$ of compact, complex manifolds over a complex analytic space $\mathcal{B}$ such that $X \cong \pi^{-1}(b)$ and $X' \cong \pi^{-1}(b')$ for some points $b, b' \in \mathcal{B}$. The manifold X is said to be a deformation in the large of X' if both are in the same equivalence class with respect to the equivalence relation generated by $\sim_{\mathrm{def}}$, which is the case if and only if both are in the same connected component of $\mathcal{C}$.

Even the seemingly natural fact that any deformation in the large of a complex torus is again a complex torus has been fully proved only in 2002 by Catanese [Cat02]. In [Cat04] he studies more in general deformations in the large of principal holomorphic torus bundles, especially bundles of elliptic curves. This was the starting point for our research.

It turns out (see [CF06]) that the right context to generalise Catanese's results is the theory of left-invariant complex structures on nilmanifolds, i.e., compact quotients of nilpotent real Lie groups by discrete subgroups.

Nilmanifolds with left-invariant complex structure provide an important source for examples in complex differential geometry. Among these are the so-called Kodaira-Thurston manifolds (see Section 1.3) which were historically the first examples of manifolds which admit both a complex structure and a symplectic structure but no Kähler structure. In fact, a nilmanifold M admits a Kähler structure if and only if it is a complex torus [BG88, Has89] and we will show in Section 1.4 that nilmanifolds can be arbitrarily far from Kähler manifolds in the sense that the Frölicher spectral sequence may be arbitrarily non degenerate.

But unfortunately, even if every (iterated) principal holomorphic torus bundle can be regarded as a nilmanifold, the converse is far from true. Moreover it turns out that even a small deformation of a principal holomorphic torus bundle may not admit such a structure (see Example 1.14).

Hence we are concerned with these three problems:

- Determine all small deformations of nilmanifolds with left-invariant complex structure.
- Give condition under which a left-invariant complex structure on a nilmanifold gives rise to a structure of (iterated) principal holomorphic torus bundle.
- Study deformations in the large of (iterated) principal holomorphic torus bundles.

There is already a vast literature concerning nilpotent Lie algebras and left-invariant complex structures on nilmanifolds (see e.g. the articles of Console, Cordero, Fernandez, Fino, Grantcharov, Gray, McLaughlin, Pedersen, Poon, Salamon, Ugarte, et al. cited in the bibliography) and we recapitulate the results in Section 1 emphasising the complex geometric structure.

Nilmanifolds will be described by a triple $M = (\mathfrak{g}, J, \Gamma)$ where $\mathfrak{g}$ is the nilpotent Lie algebra associated to a simply connected nilpotent Lie group G, J is an integrable complex structure on $\mathfrak{g}$ (see Section 1.1) and $\Gamma \subset G$ is a (cocompact) lattice. Note that the datum of either $\mathfrak{g}$ or Γ (considered as an abstract group) determines G up to unique isomorphism.

The general philosophy is that the geometry of the compact, complex manifold $M = \Gamma\backslash G$ should be completely determined by the linear algebra of $\mathfrak{g}$, J and the $\mathbb{Q}$-subalgebra generated by $\log \Gamma \subset \mathfrak{g}$.

In order to control small deformations using Kuranishi theory we have to get a good grip on the Dolbeault cohomology of the holomorphic tangent bundle.

In Section 2 we set up a Lie algebra Dolbeault cohomology with values in integrable modules (Definition 2.1) and prove an analogue of Serre Duality in this context (Theorem 2.18). Since it is known that for nilmanifolds the usual Dolbeault cohomology $H^{p,q}(M) = H^q(M, \Omega_M^p)$ can (nearly always) be calculated using invariant forms [CF01, CFGU00], this enables us to identify the cohomology of the tangent bundle with the the cohomology of the complex

$$0 \to \mathfrak{g}^{1,0} \xrightarrow{\bar\partial} \mathfrak{g}^{*0,1} \otimes \mathfrak{g}^{1,0} \xrightarrow{\bar\partial} \Lambda^2 \mathfrak{g}^{*0,1} \otimes \mathfrak{g}^{1,0} \xrightarrow{\bar\partial} \dots$$

as explained in Section 2.3.

Kuranishi theory then yields an explicit description of the small deformations:

Theorem 3.5 — *Let $M_J = (\mathfrak{g}, J, \Gamma)$ be a nilmanifold with left-invariant complex structure such that*

$$\iota : H^{1,q}((\mathfrak{g}, J), \mathbb{C}) \to H^{1,q}(M_J) \text{ is an isomorphism for all } q. \qquad (*)$$

Then all small deformations of the complex structure J are again left-invariant complex structures.

This generalises the analogous result for abelian complex structures due to Console, Fino and Poon [CFP06] (see also [MPPS06]).

There are in fact no counter-examples known for which $(*)$ does not hold and it is widely believed that the following question has a posivite answer

Question 1 — *Does $(*)$ hold for every left-invariant complex structure on a nilmanifold?*

The real geometry of nilmanifolds is well understood but only the existence of what we call a stable (complex) torus bundle series in the Lie algebra (Definition 1.10) gives us sufficient control over the complex geometry of (M, J). Geometrically this notion means the following:

On any real nilpotent Lie group G there is a filtration by subgroups, e.g., the ascending central series,

$$G \supset H_s \supset \cdots \supset H_1 \supset H_0 = \{1\}$$

such that H_k/H_{k-1} is contained in the centre of G/H_{k-1} and $\Gamma \cap H_k$ is a lattice in H_k for any lattice $\Gamma \subset G$.

In other words, any compact nilmanifold $M = \Gamma\backslash G$ can be represented as a tower of differentiable principal torus bundles $\pi_k : M_k \to M_{k+1}$

$$\begin{array}{ccc}
T_1 & \hookrightarrow & M_1 = M \\
 & & \downarrow{\scriptstyle \pi_1} \\
T_2 & \hookrightarrow & M_2 \\
 & & \downarrow{\scriptstyle \pi_2} \\
 & & \vdots \\
 & & \downarrow \\
T_{s-1} & \hookrightarrow & M_{s-1} \\
 & & \downarrow{\scriptstyle \pi_{s-1}} \\
 & & M_s = T_s
\end{array}$$

where the compact torus T_k is the quotient of H_k/H_{k-1} by the lattice and M_k is the compact nilmanifold obtained from the Lie group G/H_{k-1} by taking the quotient with respect to the image of Γ.

Such a decomposition into an iterated principal bundle is far from unique. If the subgroups can be chosen in such a way that for every left-invariant complex structure J on M we have

(*i*) The complex structure J induces a left-invariant complex structure on M_k for all k.

(*ii*) All the maps $\pi_k : M_k \to M_{k+1}$ are holomorphic with respect to these complex structures.

then we say that $\mathfrak{g}$ admits a stable complex torus bundle series.

Note that, by Theorem 3.5, this condition on the Lie algebra is stronger than to ask that (M, J) has a structure of iterated principal holomorphic torus bundle which is stable under small deformation.

This phenomenon is not so uncommon as it may seem at first sight, for example every principal holomorphic torus bundle over an elliptic curve which is not a product has this property (see Example 1.13). Section 5 will be devoted to giving a fairly complete picture of the occurring cases if the commutator $[\mathfrak{g}, \mathfrak{g}]$ has dimension at most three.

A large part of the classification of complex structures in real dimension six as obtained by Salamon and Ugarte [Sal01, Uga04] can be recovered from our more general results.

It turns out that finding a stable complex torus bundle series for some nilmanifold (M, J) is a good step on the way to prove that every deformation in the large of M is again such a nilmanifold. Indeed in this case the holomorphic fibration over a torus $\pi : M \to T_s$ can be realised as a (topologically) fixed quotient of the Albanese variety and this will enable us in Section 4 to determine all deformations in the large if the fibres of π have sufficiently nice properties:

Theorem 4.12 — *Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and let $\Gamma \subset G$ be a lattice such that the following holds:*

(*i*) *$\mathfrak{g}$ admits a stable complex torus bundle series $(\mathcal{S}^i\mathfrak{g})_{i=0,\ldots,t}$ (cf. Definition 1.10).*

(*ii*) *The nilmanifolds of the type $(\mathcal{S}^{t-1}\mathfrak{g}, J, \Gamma \cap \exp(\mathcal{S}^{t-1}\mathfrak{g}))$ constitute a good fibre class (cf. Definition 4.9).*

Then any deformation in the large M' of a nilmanifold of type $M = (\mathfrak{g}, J, \Gamma)$ is again of the same type $M' = (\mathfrak{g}, J', \Gamma)$.

Generalising the methods used in [Cat04] we will have to deal with the fact that the dimension of the Albanese variety may vary in a family of nilmanifolds.

In Section 6 we apply our results on the deformation of complex structures to the classification obtained in Section 5.

Let us denote the descending central series by $\mathcal{C}^i\mathfrak{g}$ and the centre in a Lie algebra with $\mathcal{Z}\mathfrak{g}$.

Theorem 6.1 — *Let $M = (\mathfrak{g}, J, \Gamma)$ a nilmanifold with left-invariant complex structure.*

(*i*) *Any small deformation of M is again a nilmanifold with left-invariant complex structure of the form $M' = (\mathfrak{g}, J', \Gamma)$ if one of the following conditions holds*

- $\dim \mathcal{C}^1\mathfrak{g} \leq 2$.
- $\dim \mathcal{C}^1\mathfrak{g} = 3$ *and* $\mathfrak{g}$ *is 4-step nilpotent.*
- $\dim \mathcal{C}^1\mathfrak{g} = 3$, $\mathfrak{g}$ *is 3-step nilpotent and* $\dim(\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}) = 2$.

(*ii*) *Any deformation in the large of M is again a nilmanifold with left-invariant complex structure of the form $M' = (\mathfrak{g}, J', \Gamma)$ if one of the following holds:*

- $\dim \mathcal{C}^1\mathfrak{g} \leq 1$.
- $\dim \mathcal{C}^1\mathfrak{g} = 2$, $\mathfrak{g}$ *is 2-step nilpotent and* $\dim(\mathcal{Z}\mathfrak{g})$ *is odd or equal to 2.*
- $\mathfrak{g}$ *is 3-step nilpotent and admits a stable torus bundle series of the form* $\mathfrak{g} \supset \mathcal{S}^2\mathfrak{g} \supset \mathcal{S}^1\mathfrak{g} \supset 0$ *such that* $\dim(\mathcal{C}^1(\mathcal{S}^2\mathfrak{g})) = 1$.
- ...

Moreover in all the above cases both M and M′ have the structure of an iterated principal holomorphic torus bundle with fibre dimensions depending only on $\mathfrak{g}$ *in the cases described in* (*ii*).

Note that the conditions in (*i*) are only used to ensure that (∗) in Theorem 3.5 is satisfied and if Question 1 has a positive answer (*i*) will hold for all left-invariant complex structures.

On the other hand, in order to apply Theorem 4.11 we need rather strong assumptions, e.g., the existence of a stable torus bundle series, and there are in fact examples where these are not satisfied (see Example 1.14). This leads to

Question 2 — *Which is the simplest example of a nilmanifold M_J with left-invariant complex structure such that not every deformation in the large carries a left-invariant complex structure?*

Using the classification of complex structures on real 6-dimensional, nilpotent Lie algebras ([Mag86, Sal01, Uga04], see Section 6.2 for the notation) we obtain:

Theorem 6.3 — *Let $M = (\mathfrak{g}, J, \Gamma)$ be a complex 3-dimensional nilmanifold with left-invariant complex structure. If $\mathfrak{g}$ is not in $\{\mathfrak{h}_7, \mathfrak{h}_{19}^-, \mathfrak{h}_{26}^+\}$, then M has the structure of an iterated principal holomorphic torus bundle. We list the possibilities in the following table:*

base	*fibre*	*corresponding Lie algebras*
3-torus	-	$\mathfrak{h}_1$
2-torus	*elliptic curve*	$\mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6$
elliptic curve	*2-torus*	$\mathfrak{h}_8$
Kodaira surface	*elliptic curve*	$\mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_{13}, \mathfrak{h}_{14}, \mathfrak{h}_{15}, \mathfrak{h}_{16}$

In particular the geometry is already determined by the real Lie algebra $\mathfrak{g}$. Every deformation in the large of M has the same structure.

If $\mathfrak{g} = \mathfrak{h}_7$ then there is a dense subset of the space of all left-invariant complex structures for which M admits the structure of principal holomorphic bundle of elliptic curves over a Kodaira surface but this is not true for all complex structures.

The remaining cases $\mathfrak{h}_{19}^-$ and $\mathfrak{h}_{26}^+$ do not admit the structure of principal holomorphic torus bundle for any complex structure.

In order to motivate the results in the single sections before applying them in greatest generality in Section 6 we will use the simple example of a principal holomorphic torus bundle over an elliptic curve (which is not a product) to motivate our considerations and give a model application of our theorems in every section.

We will describe the underlying Lie algebra and the stable torus bundle series in Example 1.13 and prove that every small deformation of such a manifold has the same structure in Corollary 3.6. In Example 4.4 we will describe its Albanese variety and prove that every deformation in the large is of the same type in 4.14.

Even if we had only these applications in mind our results are in fact slightly more general. The results concerning Albanese quotients apply to more general fibrations over tori and the Lie algebra Dolbeault cohomology theory could be useful in the study of the differential graded algebras arising from nilpotent Lie algebras with a complex structure (see e.g. [Poo04, And06]).

It would also be interesting to study the space of left-invariant complex structures (as defined in Section 3) more in detail, for example determine when it is smooth or universal, and perhaps look for some kind of moduli space or a description of a connected component of the Teichmüller space. These questions have already been addressed in several cases by various authors [Cat04, CF06, KS04, GMPP04].

I apologise in advance for certainly having missed some references during the text, especially since most of the theorems have been known in special cases (in particular very much was known about abelian complex structures).

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1 Nilpotent Lie algebras and nilmanifolds with left-invariant complex structure

In this section we will introduce the objects of our study and describe their basic properties. We will emphasise the complex-geometric structure of the nilmanifolds but the expert will find nothing new apart from the non collapsing result for the Frölicher spectral sequence in Section 1.4.

The geometrically important notion of stable torus bundle series will be given in Definition 1.10.

1.1 Lie algebras with a complex structure

We will throughout need the yoga of almost complex structures and will now recall some basic definitions and notations.

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and J an almost complex structure on the underlying real vector space i.e. J is an endomorphism of $\mathfrak{g}$ such that $J^2 = -Id_{\mathfrak{g}}$. This makes $\mathfrak{g}$ into a complex vector space where the multiplication with $i \in \mathbb{C}$ is given by J, the complex subspaces being exactly the real subspaces of $\mathfrak{g}$ which are invariant under the action of J. We will often switch between both points of view.

Note that the real dimension of a vector space with a complex structure is necessarily even.

As usual we denote by $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ the complexification of $\mathfrak{g}$ which yields a decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ into the $\pm i$ eigenspaces of the complex linear extension of J. The projection $\mathfrak{g} \to \mathfrak{g}^{1,0}$ is a canonical isomorphism of complex vector spaces.

Giving an almost complex structure J on $\mathfrak{g}$ is equivalent to give a complex subspace of $U \subset \mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C}} = U \oplus \bar{U}$, that is, $U \cap \bar{U} = 0$ and $2 \dim_{\mathbb{C}} U = \dim_{\mathbb{R}} \mathfrak{g}$. The real endomorphism J is then uniquely determined by the property $JX = iX$ for all $X \in U$ which means $\mathfrak{g}^{1,0} = U$.

Usually we will use small letters $x, y, \dots$ for elements of $\mathfrak{g}$ and capital letters $X, Y, \dots$ for elements in $\mathfrak{g}^{1,0}$. Elements in $\mathfrak{g}^{0,1}$ will be denoted by $\bar{X}, \bar{Y}, \dots$ using complex conjugation.

By abuse of notation we denote the complexification of linear maps with the same symbol and hope that no confusion will arise.

The exterior algebra of the dual vector space $\mathfrak{g}^*$ decomposes as

$$\Lambda^k \mathfrak{g}^* = \bigoplus_{p+q=k} \Lambda^p \mathfrak{g}^{*1,0} \otimes \Lambda^q \mathfrak{g}^{*0,1} = \bigoplus_{p+q=k} \Lambda^{p,q} \mathfrak{g}^*$$

and we have $\overline{\Lambda^{p,q}\mathfrak{g}^*} = \Lambda^{q,p}\mathfrak{g}^*$. A general reference for the linear algebra coming with a complex structure can be found in [Huy05] (Section 1.2).

Definition 1.1 — *An almost complex structure J on a real Lie algebra $\mathfrak{g}$ is said to be* ***integrable*** *if the Nijenhuis condition*

$$[x,y] - [Jx,Jy] + J[Jx,y] + J[x,Jy] = 0 \tag{1}$$

holds for all $x, y \in \mathfrak{g}$ and in this case we call the pair $(\mathfrak{g}, J)$ a Lie algebra with complex structure.

Hence by a complex structure on a Lie algebra we will always mean an integrable one. Otherwise we will speak of almost complex structures. We will mainly be concerned with nilpotent Lie algebras.

Remark 1.2 — (*i*) The real Lie algebra $\mathfrak{g}$ has the structure of a complex Lie algebra induced by J if and only if $J[x,y] = [Jx,y]$ holds for all $x, y \in \mathfrak{g}$ and J is then automatically integrable in the above sense.

(*ii*) It is easy to show that J is integrable if and only if $\mathfrak{g}^{1,0}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with the induced bracket. Note that the structure of this complex Lie algebra can be very different from the structure of $\mathfrak{g}$. For example if $\mathfrak{g}$ is nilpotent and $\dim_{\mathbb{R}}([\mathfrak{g},\mathfrak{g}]) = 1$ then the subalgebra $\mathfrak{g}^{1,0}$ is always abelian while $\mathfrak{g}$ is not (see Section 5.2).

(*iii*) If G is a real Lie group with Lie algebra $\mathfrak{g}$ then giving a left-invariant almost complex structure on G is equivalent to giving an almost complex structure J on $\mathfrak{g}$ and J is integrable if and only if it is integrable as an almost complex structure on G. It then induces a complex structure on G by the Newlander-Nirenberg theorem ([KN69], p.145) and G becomes a complex manifold. The elements of G act holomorphically by multiplication on the left but G is not a complex Lie group in general.

1.2 Nilmanifolds with left-invariant complex structure

In this section we will collect a bunch of results on nilmanifolds, most of them well know and for which we claim no originality. But our presentation will emphasise the geometric structure of compact nilmanifolds.

If not otherwise stated $(\mathfrak{g}, J)$ will always be a Lie algebra with (integrable) complex structure in the sense of (1) and G will be a associated simply connected Lie group. By a torus we will always mean a compact torus.

Definition 1.3 — *A* ***nilmanifold with left-invariant complex structure*** *is given by a triple $M_J = (\mathfrak{g}, J, \Gamma \subset G) = (\mathfrak{g}, J, \Gamma)$ where $\mathfrak{g}$ is a real nilpotent Lie algebra, J is a integrable complex structure on $\mathfrak{g}$ (in the sense of* (1) *in Section 1.1) and Γ is a lattice in a simply connected Lie group G with Lie algebra $\mathfrak{g}$.*

We will use the same letter $M_J = M$ for the compact complex manifold $\Gamma\backslash G$ endowed with the left-invariant complex structure induced by J.

A lattice in a simply connected nilpotent Lie group is a discrete cocompact subgroup. We will discuss this more in detail below.

By abuse of notation we will often omit G from the notation even if both Γ, regarded as an abstract lattice, and $\mathfrak{g}$, regarded as an abstract Lie algebra, determine G only up to canonical isomorphism.

A nilmanifold $M = (\mathfrak{g}, J, \Gamma)$ with left-invariant complex structure is Kählerian if and only if $\mathfrak{g}$ is abelian and M is a complex torus (see [BG88, Has89]). In Section 1.4 we will see that nilmanifolds can in fact be very far from Kähler manifolds.

For nilpotent Lie groups the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism and all analytic subgroups are closed and simply connected as well ([Var84], Theorem. 3.6.2, p. 196).

The following often gives the possibility to use inductive arguments.

Lemma 1.4 — *Let $(\mathfrak{g}, J)$ be a nilpotent Lie algebra with complex structure. Let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$ such that $J\mathfrak{h} = \mathfrak{h}$, i.e. $\mathfrak{h}$ is a complex subspace of the complex vector space $(\mathfrak{g}, J)$. Let G and H be the associated simply connected Lie groups endowed with the left-invariant complex structures induced by J. Then there is a holomorphic fibration $\pi : G \to G/H$ with typical fibre H.*

Proof. The map $\pi : G \to G/H$ is a real analytic fibration by the theory of Lie groups and Lie algebras since H is closed. Hence it remains to show that the differential of π is $\mathbb{C}$-linear and since the complex structure is left-invariant it suffices to do so at the identity. But here the differential is given by the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ which is $\mathbb{C}$ linear by assumption. □

Remark that we used the nilpotency of $\mathfrak{g}$ only to ensure that H is a closed subgroup.

1.2.1 The real structure of $\Gamma\backslash G$

We leave aside the complex structure for a moment and describe the geometry of the underlying real manifold. In the Lie algebra $\mathfrak{g}$ we have the following filtrations:

- The descending central series (nilpotent series) is given by

$$\mathcal{C}^0\mathfrak{g} := \mathfrak{g}, \qquad \mathcal{C}^{i+1}\mathfrak{g} := [\mathcal{C}^i\mathfrak{g}, \mathfrak{g}]$$

- The ascending central series is given by

$$\mathcal{Z}^0\mathfrak{g} := 0, \qquad \mathcal{Z}^{i+1}\mathfrak{g} := \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] \subset \mathcal{Z}^i\mathfrak{g}\}.$$

  In particular $\mathcal{Z}^1\mathfrak{g} = \mathcal{Z}\mathfrak{g}$ is the centre of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ is called *s-step nilpotent* if $\mathcal{C}^s\mathfrak{g} = 0$ and $\mathcal{C}^{s-1}\mathfrak{g} \neq 0$ or equivalently $\mathcal{Z}^s\mathfrak{g} = \mathfrak{g}$ and $\mathcal{Z}^{s-1}\mathfrak{g} \subsetneq \mathfrak{g}$.

Let G_i be the simply connected nilpotent Lie group associated to the Lie algebra $\mathfrak{g}/\mathcal{Z}^{i-1}\mathfrak{g}$ and let $F_i \cong \mathbb{R}^{n_i}$ be the abelian Lie group corresponding to $\mathcal{Z}^i\mathfrak{g}/\mathcal{Z}^{i-1}\mathfrak{g}$. Then the map $G_i \to G_{i+1}$ is a principal bundle with fibre F_i and, if $\mathfrak{g}$ is s-step nilpotent, we get a tower of such bundles:

$$\begin{array}{ccc} F_1 & \hookrightarrow & G_1 = G \\ & & \downarrow \\ F_2 & \hookrightarrow & G_2 \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ F_{s-1} & \hookrightarrow & G_{s-1} \\ & & \downarrow \\ & & G_s = F_s. \end{array}$$

Definition 1.5 — *A manifold which admits such a tower of principal bundles is called a* ***s-step iterated principal bundle****.*

In the context of nilmanifolds the fibres will always be abelian Lie groups and we will consider $\mathbb{R}^k$-bundles, real torus bundles, holomorphic $\mathbb{C}^k$-bundles and holomorphic torus bundles. Note that for us a torus is always supposed to be compact.

We will later see that the above filtration on G is compatible with any lattice Γ and hence we can describe our real manifold M as s-step iterated real torus bundle.

In general we say that the structure of iterated principal bundle is induced by a filtration $\mathcal{S}^k\mathfrak{g}$ in $\mathfrak{g}$ if we have a tower of principal bundles as above such that F_i is (the quotient by the lattice of) the Lie group corresponding to $\mathcal{S}^i\mathfrak{g}/\mathcal{S}^{i-1}\mathfrak{g}$ and G_i is (the quotient by the lattice of the) Lie group associated to the Lie algebra $\mathfrak{g}/\mathcal{S}^{i-1}\mathfrak{g}$.

1.2.2 The complex geometry of the universal covering G

Unfortunately the situation becomes much more complicated if we take the complex structure into account and we cannot hope to get always such a simple description as in the real case. This can already be seen on the level of the Lie algebra:

- The descending central series associated to J is defined by

$$\mathcal{C}^i_J\mathfrak{g} := \mathcal{C}^i\mathfrak{g} + J\mathcal{C}^i\mathfrak{g}$$

- The ascending central series associated to J (minimal torus bundle series) is defined by

$$\mathcal{T}^0\mathfrak{g} := 0, \qquad \mathcal{T}^{i+1}\mathfrak{g} := \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] \subset \mathcal{T}^i\mathfrak{g} \text{ and } [Jx, \mathfrak{g}] \subset \mathcal{T}^i\mathfrak{g}\}$$

The complex structure J is called *nilpotent* if there is a k such that $\mathcal{T}^k\mathfrak{g} = \mathfrak{g}$ and then the following properties hold:

(*i*) $\mathcal{C}^i_J\mathfrak{g}$ is a J-invariant subalgebra of $\mathfrak{g}$ and an ideal in $\mathcal{C}^{i-1}_J\mathfrak{g}$ [CF01].

(*ii*) We have always $\mathcal{C}^i\mathfrak{g} \subset \mathcal{C}^i_J\mathfrak{g}$ but in general inclusion can be strict. Nevertheless we have always $\mathcal{C}^1_J\mathfrak{g} \neq \mathfrak{g}$. ([Sal01], Theorem. 1.3, see also Example 1.14 in the next section.)

(*iii*) There is always a holomorphic fibration of G over the abelian Lie group (vector space) $\mathfrak{g}/\mathcal{C}^1_J\mathfrak{g}$ whose typical fibre is the simply connected Lie group H associated to $\mathcal{C}^1_J\mathfrak{g}$ with the left-invariant complex structure induced by the restriction of J. This is a real principal H-bundle but in general H will not be a complex Lie group so there is no way to speak of a holomorphic principal bundle.

(*iv*) Every $\mathcal{T}^i\mathfrak{g}$ is a complex subspace and an ideal of $\mathfrak{g}$. We have $\mathcal{T}^i\mathfrak{g} \subset \mathcal{Z}^i\mathfrak{g}$ for all i and if $\mathcal{T}^i\mathfrak{g} = \mathcal{Z}^i\mathfrak{g}$ then $\mathcal{T}^{i+1}\mathfrak{g}$ is the largest complex subspace of $\mathfrak{g}$ contained in $\mathcal{Z}^{i+1}\mathfrak{g}$. ([CFGU00], Lem. 3)

Not every complex structure is nilpotent, indeed there are examples (cf. 1.16) such that $\mathcal{T}^i\mathfrak{g} = 0$ for all i. In general, if $\mathfrak{g}$ is s-step nilpotent and the complex structure is nilpotent then we have $\mathcal{T}^k\mathfrak{g} = \mathfrak{g}$ for some $k \geq s$ and strict inequality is possible. (See [CFGU00] or Example 1.14)

Cordero, Fernandez, Gray and Ugarte showed the following (see e.g. [CFGU00]):

Proposition 1.6 — *Let $(\mathfrak{g}, J)$ be a nilpotent Lie algebra with complex structure. Then J is nilpotent if and only if the associated simply connected Lie group G has the structure of a k-step iterated $\mathbb{C}^{n_i}$-bundle where k is the smallest integer such that $\mathcal{T}^k\mathfrak{g} := \mathfrak{g}$. The structure of iterated bundle is induced by the filtration $(\mathcal{T}^i\mathfrak{g})$ on the Lie algebra.*

Warning: The nilpotency of a complex structure J is a necessary condition for the nilmanifold $M = (\mathfrak{g}, J, \Gamma)$ to admit the structure of iterated principal holomorphic torus bundle but it is by no means sufficient. We will discuss this below and in Example 1.14.

Let X be an element in $\mathfrak{g}^{1,0}$. Then the map $ad_X = [X, -] : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ can be decomposed into its components with respect to the decomposition

$$\mathrm{Hom}(\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}) = \mathrm{Hom}(\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}) \oplus \mathrm{Hom}(\mathfrak{g}^{1,0}, \mathfrak{g}^{0,1}) \oplus \mathrm{Hom}(\mathfrak{g}^{0,1}, \mathfrak{g}_\mathbb{C})$$

We write $ad_X = A + B + C$. The component B vanishes for all X since J is integrable.

The complex structure is called *abelian* if $A = 0$ for all X which is equivalent to say that $\mathfrak{g}^{1,0}$ is an abelian subalgebra of $\mathfrak{g}_{\mathbb{C}}$ which is in turn equivalent to $[x, y] = [Jx, Jy]$ for all $x, y \in \mathfrak{g}$ (see also Lemma 5.1). This implies that $J\mathcal{Z}^i\mathfrak{g} = \mathcal{Z}^i\mathfrak{g}$ for all i and in particular every abelian complex structure is nilpotent with $\mathcal{T}^i\mathfrak{g} = \mathcal{Z}^i\mathfrak{g}$. These structure were extensively studied by Salamon, McLaughlin, Pedersen, Poon, Console, Fino e.a. for example in [CFP06] and [MPPS06].

If C is zero for all $X \in \mathfrak{g}^{1,0}$ then G is complex parallelisable. These manifolds have been described in detail in by Winkelmann [Win98] and have very special (arithmetic) properties.

1.2.3 The complex geometry of $M = \Gamma\backslash G$

We will now address the question of the compatibility of the lattice $\Gamma \subset G$ with the other two structures $\mathfrak{g}$ and J. Most of the cited results originate from the work of Malcev [Mal51].

Definition 1.7 — *Let $\mathfrak{g}$ be a nilpotent Lie algebra. A* ***rational structure*** *for $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_{\mathbb{Q}}$ defined over the rationals such that $\mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R} = \mathfrak{g}$.*

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be rational with respect to a given rational structure $\mathfrak{g}_{\mathbb{Q}}$ if $\mathfrak{h}_{\mathbb{Q}} := \mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q}}$ is a rational structure for $\mathfrak{h}$.

If Γ is a lattice in the corresponding simply connected Lie group G then its associated rational structure is given by the $\mathbb{Q}$-span of $\log \Gamma$. A rational subspace with respect to this structure is called ***Γ-rational****.*

By a lattice in the Lie algebra $\mathfrak{g}$ we mean a lattice in the underlying vector space which is closed under bracket and we say that $\Gamma \subset G$ is induced by a lattice in $\mathfrak{g}$ if $\log \Gamma$ is a lattice in $\mathfrak{g}$.

Remark 1.8 — (*i*) The rational structure associated to Γ is in fact a rational structure ([CG90], p. 204). In particular there exists a lattice in a nilpotent simply connected Lie group G if and only if the corresponding Lie algebra admits a rational structure.

(*ii*) If $\Gamma \subset G$ is a lattice then there exist lattices $\Gamma_1 \subset \Gamma_2 \subset \mathfrak{g}$ such that Γ_1 is of finite index in Γ_2 and $\exp \Gamma_1 \subset \Gamma \subset \exp \Gamma_2$. ([VGS00], Theorem. 2.13, p. 48) Geometrically this corresponds to taking a finite étale cover (resp. quotient) of the nilmanifold $\Gamma\backslash G$.

(*iii*) If two lattices $\Gamma_1, \Gamma_2 \subset G$ generate the same rational structure then the lattice $\Gamma_1 \cap \Gamma_2$ is of finite index in both of them ([CG90], Theorem

5.1.12, p. 205). Geometrically we have an étale correspondence

$$\begin{array}{ccc} & (\Gamma_1 \cap \Gamma_2)\backslash G & \\ \swarrow & & \searrow \\ \Gamma_1\backslash G & & \Gamma_2\backslash G. \end{array}$$

(*iv*) The subalgebras $\mathcal{C}^i\mathfrak{g}$ and $\mathcal{Z}^i\mathfrak{g}$ as defined above are in fact always rational subalgebras. ([CG90], p. 208) In particular $\Gamma\backslash G$ has the structure of real analytic iterated principal torus bundle induced by the filtration $(\mathcal{Z}^i\mathfrak{g})$ on the Lie algebra.

Lemma 1.9 — *If in the situation of Lemma 1.4 $\Gamma \subset G$ is a lattice then π induces a fibration on the compact nilmanifold $M = \Gamma\backslash G$ if and only if $\Gamma \cap H$ is a lattice in H if and only if the associated subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is Γ-rational.*

In particular M fibres as a principal holomorphic torus bundle $\pi : M \to M'$ over some nilmanifold M' if and only if there is a J-invariant, Γ-rational subspace contained in the centre $\mathcal{Z}\mathfrak{g}$ of $\mathfrak{g}$.

Proof. The first part is [CG90], Theorem. 5.1.11, p. 204 and Lem 5.1.4, p. 196. The second assertion then follows. □

The minimal torus bundle series as defined above is not necessarily the best choice to study nilmanifolds as will be explained in Example 1.15.

Definition 1.10 — *Let $\mathfrak{g}$ be a nilpotent Lie algebra. We call an ascending filtration $(\mathcal{S}^i\mathfrak{g})_{i=0,\dots,t}$ on $\mathfrak{g}$ a* ***(complex) torus bundle series*** *for a complex structure J if*

$$\begin{gathered} \mathcal{S}^0\mathfrak{g} = 0, \qquad \mathcal{S}^t\mathfrak{g} = \mathfrak{g}, \\ J\mathcal{S}^i\mathfrak{g} = \mathcal{S}^i\mathfrak{g} \qquad (i = 0, \dots, t), \\ \mathcal{S}^{i+1}\mathfrak{g}/\mathcal{S}^i\mathfrak{g} \subset \mathcal{Z}(\mathfrak{g}/\mathcal{S}^i\mathfrak{g}) \qquad (i = 0, \dots, t). \end{gathered}$$

An ascending filtration $(\mathcal{S}^i\mathfrak{g})_{i=0,\dots,t}$ on $\mathfrak{g}$ is said to be a ***stable (complex) torus bundle series*** *on $\mathfrak{g}$, if $(\mathcal{S}^i\mathfrak{g})_{i=0,\dots,t}$ is a torus bundle series for every complex structure J on $\mathfrak{g}$ and every subspace $\mathcal{S}^i\mathfrak{g}$ is rational with respect to every lattice in the simply connected Lie group associated to $\mathfrak{g}$.*

We will usually omit the word complex and speak simply of stable torus bundle series.

The rationality condition holds for example if every subspace in the series can be described by the subspaces of the ascending and descending central series. Note that the conditions imply that the subspaces $\mathcal{S}^i\mathfrak{g}$ are in fact ideals in $\mathfrak{g}$ and that J is a nilpotent complex structure if and only if $\mathfrak{g}$ admits a torus bundle series with respect to J if and only if G has the structure of iterated $\mathbb{C}^k$-bundle.

Remark 1.11 — If $\mathfrak{g}$ admits a stable torus bundle series then every nilmanifold $M = (\mathfrak{g}, J, \Gamma)$ has the structure of iterated principal holomorphic torus bundle. More precisely let

$$\begin{array}{ccc} T_1 & \hookrightarrow & M_1 = M \\ & & \downarrow{\scriptstyle \pi_1} \\ T_2 & \hookrightarrow & M_2 \\ & & \downarrow{\scriptstyle \pi_2} \\ & & \vdots \\ & & \downarrow \\ T_{s-1} & \hookrightarrow & M_{t-1} \\ & & \downarrow{\scriptstyle \pi_{t-1}} \\ & & M_t = T_t. \end{array}$$

be the structure of real analytic iterated principal torus bundle on M which corresponds to the stable torus bundle filtration $(\mathcal{S}^i\mathfrak{g})_{i=0,\ldots,t}$. Then the maps π_i are in fact holomorphic with respect to any left-invariant complex structure J, since the differentials at the identity are given by the complex linear maps

$$\mathcal{S}^i\mathfrak{g}/\mathcal{S}^{i-1} \to \mathfrak{g}/\mathcal{S}^{i-1}.$$

A stable torus bundle series gives in some sense the appropriate description of geometry of the corresponding nilmanifolds. This will become clearer when we study some examples in Section 1.3.

In general a stable torus bundle series will not exist in a Lie algebra. For example the Lie algebra $\mathfrak{h}_7$ in 1.14 does not admit a stable torus bundle series even if every complex structure on $\mathfrak{h}_7$ is nilpotent. An example of a stable torus bundle series is given in 1.13.

We will also use the following notion.

Definition 1.12 — *Let $\mathfrak{g}$ be a nilpotent Lie algebra with rational structure $\mathfrak{g}_\mathbb{Q}$ and let J be a complex structure on $\mathfrak{g}$. We say J is a* ***rational complex structure*** *if it maps $\mathfrak{g}_\mathbb{Q}$ to itself.*

It was used in [CF01] that for a rational complex structure J the subspace $\mathcal{C}^i_J\mathfrak{g}$ is a rational subspace and therefore $\mathcal{C}^1_J\mathfrak{g}$ induces a holomorphic fibration of $M = \Gamma\backslash G$ over a complex torus with typical fibre the compact complex nilmanifold $(\mathcal{C}^1_J\mathfrak{g}, J|_{\mathcal{C}^1_J\mathfrak{g}}, \Gamma \cap \mathcal{C}^1_J\mathfrak{g})$. But even if $\mathcal{C}^1_J\mathfrak{g}$ is an abelian subalgebra this fibration is not a principal bundle if $\mathcal{C}^1_J\mathfrak{g}$ is not contained in the centre of $\mathfrak{g}$. An example which exhibits this phenomenon is the Lie algebra $(\mathfrak{g}, J_2)$ described in Example 5.12.

1.3 Examples and Counterexamples

In this section we will collect a number of examples in order to illustrate some phenomena that can occur. Several other examples will be given in Section 5.

The simplest example is certainly a complex torus which corresponds to an abelian Lie algebra. But note that it is possible to deform a product of two elliptic curves to a simple torus which shows that in general we cannot hope that our favourite J-invariant, rational subspace of $\mathfrak{g}$ is still invariant after deformation.

Other well known examples are Kodaira surfaces (Kodaira-Thurston manifolds) which were historically the first manifolds shown to admit both a symplectic structure and a complex structure but no Kähler structure. These are principal bundles of elliptic curves over elliptic curves.

We will consider a slightly more general class, which we will carry along to illustrate the theory in all chapters while the more general applications will be delayed until Section 6.

Example 1.13 — A complex manifold M of dimension n, which is not a torus, admits a structure of a principal holomorphic torus bundle over an elliptic curve if and only if M is a nilmanifold with left-invariant complex structure $M = (\mathfrak{g}, J, \Gamma)$ such that the centre of $\mathfrak{g}$ has (real) codimension two in $\mathfrak{g}$.

In particular this is a property of the underlying real Lie algebra $\mathfrak{g}$ and

$$\mathfrak{g} \supset \mathcal{Z}\mathfrak{g} \supset 0$$

is a stable torus bundle series for $\mathfrak{g}$.

Proof. We already described in Section 1.2 that any principal holomorphic torus bundle can be regarded as a nilmanifold with left-invariant complex structure $(\mathfrak{g}, J, \Gamma)$. We have then a central extension

$$0 \to \mathcal{T}^1\mathfrak{g} \to \mathfrak{g} \to \mathfrak{e} \to 0$$

where $\mathcal{T}^1\mathfrak{g}$ is the $2n-2$ dimensional real, J-invariant subspace corresponding to the fibres of the bundle. In particular the centre of $\mathfrak{g}$ has dimension at least $2n-2$ but since M is not a torus, i.e. $\mathfrak{g}$ is not abelian, the centre is in fact equal to $\mathcal{T}^1\mathfrak{g}$ and has real codimension two as claimed.

Vice versa assume we have an arbitrary nilmanifold with left-invariant complex structure $M = (\mathfrak{g}, J, \Gamma)$ such that the centre has (real) codimension two in $\mathfrak{g}$. Note image of the bracket $\mathcal{C}^1\mathfrak{g}$ is a real one dimensional subspace since any subspace complementary to the centre has dimension two.

We claim that J is in fact always an abelian complex structure. Assume the contrary: let $C = [X, Y]$ be a nontrivial element in $\mathcal{C}^1\mathfrak{g}^{1,0} \subset \mathcal{C}^1\mathfrak{g}_{\mathbb{C}}$. Then $\bar{C} = [\bar{X}, \bar{Y}] \in \mathfrak{g}^{0,1}$ is another, linear independent element in $\mathcal{C}^1\mathfrak{g}_{\mathbb{C}}$ since

$\mathfrak{g}^{1,0} \cap \mathfrak{g}^{0,1} = 0$. Hence $\mathcal{C}^1\mathfrak{g}_\mathbb{C} = (\mathcal{C}^1\mathfrak{g}) \otimes \mathbb{C}$ has complex dimension at least 2 which yields a contradiction.

For an abelian complex structures the ascending central series is in fact a series of J invariant subspaces, in particular the centre is a complex subspace. Furthermore $\Gamma \cap \mathcal{C}^1\mathfrak{g}$ is in fact a lattice by [VGS00] (cor. 2.5, p. 44) and the filtration is compatible with Γ. So M has the structure of a principal holomorphic torus bundle over an elliptic curve. □

A detailed study of complex structures on Lie algebras with 1-dimensional commutator can by found in section 5.2.

In order to give further examples we have to explain some notation. Consider a Lie algebra $\mathfrak{g}$ spanned by a basis $e_1, \ldots, e_n$. Then the Lie bracket is uniquely determined by structure constants a_{ij}^k such that

$$[e_i, e_j] = \sum_{k=1}^{n} a_{ij}^k e_k$$

satisfying $a_{ij}^k = -a_{ji}^k$ and the relations encoding the Jacobi identity. Let $\langle e^1, \ldots, e^n \rangle$ be the dual basis, i.e. $e^i(e_j) = \delta_{ij}$. Then for any $\alpha \in \mathfrak{g}^*$ and $x, y \in \mathfrak{g}$ we define

$$d : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*, \qquad d\alpha(x, y) := -\alpha([x, y])$$

and get a dual description of the Lie bracket by

$$de^k = -\sum_{i<j} a_{ij}^k e^{ij}$$

where we abbreviate $e^i \wedge e^j = e^{ij}$.

The map d induces a map on the exterior algebra $\Lambda^*\mathfrak{g}^*$ and $d^2 = 0$ is equivalent to the Jacobi identity (see Section 2.3).

Sometimes we will use a notation like $\mathfrak{h}_2 = (0, 0, 0, 0, 12, 34)$ by which we mean the following: Let $e_1, \ldots, e_6$ be a basis for the Lie algebra and $e^1, \ldots, e^6$ be the dual basis. Then the defining relations for $\mathfrak{h}_2$ are given by

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{12} = e^1 \wedge e^2, \quad de^6 = e^{34}.$$

In other words the bracket relations are generated by $[e_1, e_2] = -e_5$ and $[e_3, e_4] = -e_6$. (The name $\mathfrak{h}_2$ is due to the classification of six-dimensional real Lie algebras obtained in [Sal01] which will be used in Section 6.2.)

There are several ways to describe a complex structure on a given Lie algebra. Either we give an endomorphism J on the basis of $\mathfrak{g}$ such that $J^2 = -id_\mathfrak{g}$ or we can give a complex subalgebra $\mathfrak{h} \subset \mathfrak{g}_\mathbb{C}$ such that $\mathfrak{h} \cap \bar{\mathfrak{h}} = 0$ and $\mathfrak{h} \oplus \bar{\mathfrak{h}} = \mathfrak{g}_\mathbb{C}$.

Yet another method is to start with a complex vector space U defining $\mathfrak{g}_\mathbb{C}^* := U \oplus \bar{U}$. This yields a real vector space $\mathfrak{g}$ together with a complex structure such that $\mathfrak{g}^{*1,0} = U$.

Let $d : U \to \Lambda^2\mathfrak{g}_{\mathbb{C}}^*$ be complex linear map and consider

$$\delta := d \oplus \bar{d} : \mathfrak{g}_{\mathbb{C}}^* \to \Lambda^2\mathfrak{g}_{\mathbb{C}}^*.$$

The map δ defines a Lie bracket on $\mathfrak{g}_{\mathbb{C}}$ if $\delta^2 = 0$ and then restricts by construction to a real Lie algebra structure on $\mathfrak{g}$. If we choose d in such a way that U is mapped to zero in $\Lambda^2\bar{U}$ then the complex structure on $\mathfrak{g}$ is in fact integrable. (See Section 1.4 for an example which is constructed in this way.)

We will always give a minimal number of defining relations, e.g., if $Je_1 = e_2$ then $Je_2 = -e_1$ is tacitly understood.

Note that giving a 2-step Lie algebra is the same as giving an alternating bilinear form on $\mathfrak{g}/\mathcal{Z}\mathfrak{g}$ with values in $\mathcal{Z}\mathfrak{g}$. The Jacobi Identity has not to be checked, since it is trivial.

Example 1.14 — In the next example we will see that a nice behaviour of the complex structure on the universal covering G may not be sufficient to get a nice description the manifold $M = \Gamma \setminus G$.

Consider the Lie group H_7 whose Lie algebra is $\mathfrak{h}_7 = (0, 0, 0, 12, 13, 23)$ with basis $e_1, \ldots, e_6$. The vectors $e_4 \ldots, e_6$ span the centre $\mathcal{Z}^1\mathfrak{h}_7 = \mathcal{C}^1\mathfrak{h}_7$.

Let $\Gamma \subset H_7$ be the lattice generated the elements $\exp(e_k)$ and consider the nilmanifold $M = \Gamma\backslash H_7$, which – as a real manifold – can be regarded as a real principal torus bundle with fibre and base a 3-dimensional torus.

For every real number λ we give a left-invariant complex structure J_λ on M by specifying a basis for the $(1, 0)$ part of $\mathfrak{h}_7$:

$$(\mathfrak{h}_7{}^{1,0})_\lambda := \langle X_1 = e_1 - ie_2, X_2^\lambda = e_3 - i(e_4 - \lambda e_1), X_3^\lambda = -e_5 - \lambda e_4 + ie_6\rangle$$

One can check that $[X_1, X_2^\lambda] = X_3^\lambda$ and, since X_3^λ is contained in the centre, this complex structure is in fact integrable. The largest complex subspace of the centre is spanned by the real and imaginary part of X_3^λ since the centre has real dimension three.

The simply connected Lie group H_7 has now a filtration by subgroups induced by the filtration

$$\mathfrak{h}_7 \supset V_1 = \langle \lambda e_2 + e_3, e_4, Im(X_3^\lambda), Re(X_3^\lambda)\rangle \supset V_2 = \langle Im(X_3^\lambda), Re(X_3^\lambda)\rangle \supset 0$$

on the Lie algebra and, since all these are J invariant, H_7 has the structure of a tower of principal holomorphic bundles with fibre $\mathbb{C}$. This means that the complex structure is nilpotent.

In fact, using the results of [Uga04] a simple calculation shows that essentially every complex structure on $\mathfrak{h}_7$ is equivalent to J_0. But the same does not hold any more if we take the lattice into account.

Note that the rational structure induced by Γ coincides with the $\mathbb{Q}$-algebra generated by the basis vectors e_k. Therefore $V_j \cap \Gamma$ is a lattice in V_j if and only if λ is rational. That is, for $\lambda \notin \mathbb{Q}$ the structure of iterated holomorphic principal bundle on H_7 does not descend to the quotient $\Gamma\backslash H_7$.

Example 1.15 — The aim of this example is to show that in order to understand the geometry of a nilmanifold with left-invariant complex structure it is can be useful not to consider the minimal torus bundle series $\mathcal{T}^i\mathfrak{g}$ but other torus bundle series.

We consider the Lie algebra given by $\mathfrak{g} = (0, 0, 0, 0, 0, 0, 12, 13)$ with the lattice Γ generated by the images of the basis vectors e_k.

Let $\langle z_1, z_2, z_3\rangle$ be any basis of the subspace spanned by $e_4, \dots, e_6$

We define a complex structure by declaring

$$\begin{aligned} Je_2 &= e_3, & Je_7 &= e_8, \\ Jz_1 &= e_1, & Jz_2 &= z_3. \end{aligned}$$

A short calculation shows that J is integrable and that $\mathcal{T}^1\mathfrak{g} = \langle z_2, z_3, e_7, e_8\rangle$ is the largest complex subspace of the centre. But the z_i were quite arbitrarily chosen and hence $\mathcal{T}^1\mathfrak{g}$ is not a Γ-rational subspace in general, which means that it does not tell us a lot about the geometry of the compact manifolds $M = \Gamma \setminus G$. But on the other hand $\mathcal{C}^1\mathfrak{g} = \langle e_7, e_8\rangle$ is a Γ-rational, J-invariant subspace. This is no coincidence since we will prove in Proposition 5.6 that

$$0 \subset \mathcal{C}^1\mathfrak{g} \subset \mathfrak{g}$$

is a stable torus bundle series for $\mathfrak{g}$ which means that we can describe M as a principal bundle of elliptic curves over a complex 3-torus for any complex structure on $\mathfrak{g}$.

Example 1.16 — We return to to the issue of nilpotent complex structures. It has been shown in [Sal01] that the six-dimensional Lie algebra $\mathfrak{h}_{26}^+ = (0, 0, 12, 13, 23, 14 + 25)$ does admit complex structures but since the centre has dimension one none of these structures can possibly be nilpotent.

A Lie algebra which admits both nilpotent and non nilpotent complex structures is described in Example 5.12 and in [CFP06] Console, Fino and Poon construct an example which shows that even a small deformation of an abelian complex structure may be non-nilpotent.

1.4 The Frölicher Spectral Sequence for Torus bundles

Let X be a complex manifold and $\mathcal{A}^{p,q}(X)$ be the space of smooth differential (p, q) forms. The Frölicher Spectral Sequence of X is the spectral sequence derived from the double complex $(\mathcal{A}^{p,q}(X), \partial, \overline{\partial})$ and $(E_n^{p,q}, d_n) \Rightarrow H^*(X, \mathbb{C})$. It is well known that it degenerates at the E_1 term if X is a Kähler manifold and hence a non degenerating Frölicher spectral sequence measures in some sense how far a manifold is from being a Kähler manifold.

The historically first example for this phenomenon was the Iwasawa manifold for which $E_1 \ncong E_2 \cong E_\infty$. In the book of Griffith and Harris [GH78] is repeated the question if there were examples with nonzero differential in higher terms E_n and in [CFUG99] several complex 3 dimensional cases were analysed which collapsed only at the E_3 term.

The aim of the section is to describe a family of examples which show that principal holomorphic torus bundles and hence nilmanifolds can be arbitrarily far from being Kähler manifolds in the sense that the Frölicher spectral sequence can be arbitrarily non degenerate. This is already true for 2-step nilpotent Lie algebras:

Example 1.17 — We give for n at least 2 a Lie algebra $\mathfrak{g}_n$ with complex structure defined by the following structure equations. Let

$$\{\omega_1, \dots, \omega_n, \lambda_1, \dots, \lambda_n, \eta_1 \dots, \eta_{n-1}\}$$

be a basis for ${\mathfrak{g}_n^*}^{1,0} := {\mathfrak{g}_{n\mathbb{C}}^*}^{1,0}$, the (1,0) part of the dual of $\mathfrak{g}_n$ and consider the differential $d : \mathfrak{g}_{n\mathbb{C}}^* \to \Lambda^2 \mathfrak{g}_{n\mathbb{C}}^*$ given by

$$\begin{aligned} d\omega_1 &= \bar{\lambda}_1 \wedge \eta_1 \\ d\omega_k &= \lambda_1 \wedge \bar{\eta}_{k-1} + (-1)^n \lambda_1 \wedge \lambda_k \qquad k = 2, \dots, n \\ d\eta_k &= d\lambda_k = 0 \qquad \text{for all } k \end{aligned}$$

and by the complex conjugate equations. Obviously $d^2 = 0$ and hence the Jacobi identity holds for the associated real Lie algebra structure. The Lie algebra $\mathfrak{g}_n$ is 2-step nilpotent and the centre, given by the subspace annihilated by the η_k's and λ_k's and their complex conjugates, is a complex subspace with respect to this complex structure.

Let G_n be the associated simply-connected nilpotent Lie group and let A_n be the abelian subgroup corresponding to the centre of $\mathfrak{g}_n$. Since the structure constants are integers we can choose a lattice $\Gamma \subset G_n$ such that the nilmanifold $M_n = \Gamma \backslash G_n$ is a principal torus bundle over a torus.

The differential on the exterior algebra $\Lambda^* \mathfrak{g}_{\mathbb{C}}^*$ induced by d can be decomposed in the usual way as $d = \partial + \overline{\partial}$. Considering elements in $\mathfrak{g}_{\mathbb{C}}^*$ as left-invariants differential forms on M we get an inclusion of differential bigraded algebras

$$\Lambda^{*,*} \mathfrak{g}_{\mathbb{C}}^* \hookrightarrow \mathcal{A}^{*,*}(M_n).$$

It was proved in [CFUG99] that this inclusion induces for principal holomorphic torus bundles an isomorphism of spectral sequences after the E_1 term.

Proposition 1.18 — *The Frölicher spectral sequence of M_n does not degenerate at the E_n term. More precisely we have*

$$d_n([\bar{\omega}_1 \wedge \bar{\eta}_2 \wedge \dots \wedge \bar{\eta}_{n-1}]) = [\lambda_1 \wedge \dots \wedge \lambda_n] \neq 0,$$

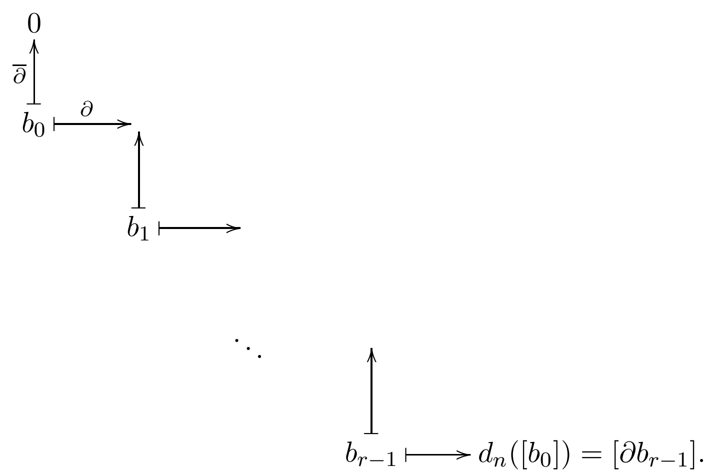
i.e., the map $d_n : E_n^{0,n-1} \to E_n^{n,0}$ is not trivial.

Proof. The E_0 term of the first quadrant spectral sequence associated to the double complex $(\Lambda^{\bullet,\bullet}\mathfrak{g}^*_{n\mathbb{C}}, \partial, \overline{\partial})$ is given by $E_0^{p,q} = \Lambda^p \mathfrak{g}_n^{*\,1,0} \otimes \Lambda^q \mathfrak{g}_n^{*\,0,1}$ and $d_0 = \overline{\partial}$. The differential d_1 is induced by ∂.

Following the exposition in [BT82] (§14, p.161ff) we say that an element of $b_0 \in E_0^{p,q}$ lives to E_r if it represents a cohomology class in E_r or equivalently if it is a cocycle in $E_0, E_1, \ldots, E_{r-1}$. This is shown to be equivalent to the existence of a *zig-zag* of length r, that is, a collection of elements $b_1, \ldots, b_{r-1}$ such that

$$b_i \in E_0^{p+i,q-i} \qquad \overline{\partial} b_0 = 0 \qquad \partial b_{i-1} = \overline{\partial} b_i (i = 1, \ldots r-1).$$

These can be represented as



In this picture we have the first quadrant double complex given by $(E_0^{p,q}, \partial, \overline{\partial})$ in mind in which this zig-zag lives.

Furthermore $d_n([b_0]) = [\partial b_{r-1}]$ is zero in $E_r^{p+n,q-n+1}$ if and only if there exists an element $b_r \in E_0^{p+n,q-n}$ such that $\overline{\partial} b_r = \partial b_{r-1}$, i.e. we can extend the zigzag by one element.

We will now show that $b_0 := \bar{\omega}_1 \wedge \bar{\eta}_2 \wedge \cdots \wedge \bar{\eta}_{n-1}$ admits a zigzag of length n which cannot be extended. Since $\overline{\partial}\bar{\omega}_1 = 0$ we have $\overline{\partial} b_0 = 0$ and calculate $\partial b_0 = \lambda_1 \wedge \bar{\eta}_1 \wedge \cdots \wedge \bar{\eta}_{n-1}$. Let us define:

$$\begin{aligned} b_1 &:= \omega_2 \wedge \bar{\eta}_2 \wedge \cdots \wedge \bar{\eta}_{n-1} \\ b_i &:= \omega_{i+1} \wedge \bar{\eta}_{i+1} \wedge \cdots \wedge \bar{\eta}_{n-1} \wedge \lambda_2 \wedge \cdots \wedge \lambda_i \qquad (i = 2, \ldots, n-2) \\ b_{n-1} &:= \omega_n \wedge \lambda_2 \wedge \cdots \wedge \lambda_{n-1} \end{aligned}$$

Then $\partial b_{i-1} = \overline{\partial} b_i (i = 1, \ldots n-1)$ and we have the desired zigzag. We conclude by saying that $d_n([b_0]) = [\partial b_{n-1}] = [\lambda_1 \wedge \ldots \lambda_n] \neq 0$ because $E_0^{n,-1} = 0$ and the zig-zag cannot be extended. □

Remark 1.19 — Note that exchanging every λ_i by η_i in the above definition of $\mathfrak{g}_n$ would yield an example of dimension $2n$ with the same properties; but we found it graphically more convincing to replace one of the $\bar{\eta}_i$'s by one of the λ_i's in every step of the zigzag in the proof. This yields

Corollary 1.20 — *For $n \geq 2$ there exist $2n$-dimensional 2-step nilmanifolds with left-invariant complex structure, which are principal holomorphic torus bundles over a base of dimension n with fibre of dimension n, such that the Frölicher spectral sequence does not degenerate at the E_n term.*

2 Lie algebra Dolbeault cohomology

The aim of this section is to set up some Dolbeault cohomology theory for modules over Lie algebras with complex structure and prove Serre duality in this context. Our main application is the calculation of the cohomology groups of the tangent sheaf for nilmanifolds with a left-invariant complex structure in Section 3 but perhaps the notions introduced are of independent interest.

In the first subsection we define the notion of (anti-)integrable module and derive the elementary properties which are interpreted in geometric terms in section two. The third section is again devoted to algebra when we set up our machinery of Lie algebra Dolbeault cohomology while the fourth one will explain the geometric implications of our theory.

2.1 Integrable representations and modules

For the whole section let $(\mathfrak{g}, J)$ be a Lie algebra with complex structure. Several times we will refer to the Nijenhuis tensor (1) which was defined in Section 1.

A left $\mathfrak{g}$-module structure on a vector space E is given by a bilinear map

$$\mathfrak{g} \times E \to E \qquad (x, v) \mapsto x \cdot v$$

such that $[x, y]\cdot v = x\cdot y\cdot v - y\cdot x\cdot v$. Note that this induces a map $\mathfrak{g} \to \operatorname{End} E$, a representation of $\mathfrak{g}$ on E. If we want to stress the Lie algebra structure on $\operatorname{End} E$ (induced by the structure of associative algebra by setting $[a, b] = ab - ba$) we use the notation $\mathfrak{gl}(E)$. A representation or a left module structure correspond hence to a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(E)$.

In the sequel we want to combine these notions with complex structures both on $\mathfrak{g}$ and E.

Let $(\mathfrak{g}, J)$ be a Lie algebra with (integrable) complex structure and E a real vector space with (almost) complex structure J'.

Definition 2.1 — *A representation $\rho : (\mathfrak{g}, J) \to \operatorname{End} E$ of $\mathfrak{g}$ on E is said to be **integrable** if for all $x \in \mathfrak{g}$ the endomorphism of E given by*

$$\mathcal{N}(x) := [J', (\rho \circ J)(x)] + J'[\rho(x), J']$$

*vanishes identically. In this case we say that (E, J') is an **integrable** $(\mathfrak{g}, J)$**-module**. We say that (E, J') is **antiintegrable** if $(E, -J')$, the complex conjugated module, is an integrable $\mathfrak{g}$-module. A homomorphism of (anti-) integrable $\mathfrak{g}$-modules is a homomorphism of underlying $\mathfrak{g}$-modules that is $\mathbb{C}$ linear with respect to the complex structures.*

The definition is motivated by the fact that the adjoint representation $ad : \mathfrak{g} \to \operatorname{End} \mathfrak{g}$ given by $x \mapsto [x, -]$ is integrable in this sense if and only if the Nijenhuis tensor (1) vanishes, i.e., if $(\mathfrak{g}, J)$ is a Lie algebra with complex structure. This is a special case of the next result.

Proposition 2.2 — *Let (E, J') be vector space with complex structure and $\rho : \mathfrak{g} \to \operatorname{End} E$ a representation. Then the following are equivalent:*

(*i*) *ρ is integrable.*

(*ii*) *For all $X \in \mathfrak{g}^{1,0}$ the map $\rho(X)$ has no component in $\operatorname{Hom}(E^{1,0}, E^{0,1})$.*

(*iii*) *$E^{1,0}$ is an invariant subspace under the action of $\mathfrak{g}^{1,0}$.*

(*iv*) *$\rho|_{\mathfrak{g}^{1,0}}$ induces a complex linear representation on $E^{1,0}$ by restriction.*

Proof. The restriction of ρ to $\mathfrak{g}^{1,0}$ is $\mathbb{C}$ linear by definition since it is the complexification of the real representation restricted to a complex subspace. Therefore condition (*ii*) is equivalent to (*iii*) and (*iv*).

It remains to prove (*i*)$\Leftrightarrow$ (*ii*).

Let $X \in \mathfrak{g}^{1,0}$ and $V \in V^{1,0}$. Using $JX = iX$ and $J'V = iV$ we calculate

$$\begin{aligned}\mathcal{N}(X)V &= ([J', (\rho \circ J)(X)] + J'[\rho(X), J'])(V)\\ &= (iJ'\rho(X) - i\rho(X)J' + J'\rho(X)J' - J'^2\rho(X))(V)\\ &= 2iJ'\rho(X)V + 2\rho(X)V\end{aligned}$$

and see that

$$\mathcal{N}(X)V = 0 \Leftrightarrow J'\rho(X)V = i\rho(X)V \Leftrightarrow \rho(X)V \in V^{1,0}.$$

This proves (*i*)$\Rightarrow$ (*ii*). Vice versa assume that (*ii*) holds. We decompose the elements in $E_\mathbb{C}$ respectively in $\mathfrak{g}_\mathbb{C}$ into their $(1,0)$ and $(0,1)$ parts. By the above calculation and its complex conjugate (the representation and hence the bracket are real and commute with complex conjugation) it remains to consider the *mixed* case. We have for all X, V as above

$$\begin{aligned}\mathcal{N}(X)\bar V &= (iJ'\rho(X) - i\rho(X)J' + J'\rho(X)J' - J'^2\rho(X))(\bar V)\\ &= iJ'\rho(X)\bar V - \rho(X)\bar V - iJ'\rho(X)\bar V + -\rho(X)\bar V\\ &= 0\end{aligned}$$

and hence ρ is integrable. □

Corollary 2.3 — *Let (E, J') be vector space with complex structure and $\rho : \mathfrak{g} \to \operatorname{End} E$ a representation. Then the following are equivalent:*

(*i*) *ρ is antiintegrable.*

(*ii*) *For all $\bar X \in \mathfrak{g}^{0,1}$ the map $\rho(\bar X)$ has no component in $\operatorname{Hom}(E^{1,0}, E^{0,1})$.*

(*iii*) *$E^{1,0}$ is an invariant subspace under the action of $\mathfrak{g}^{0,1}$.*

(*iv*) *$\rho|_{\mathfrak{g}^{0,1}}$ induces a complex linear representation on $E^{1,0}$ by restriction.*

Proof. Exchange J' by $-J'$ in the above proof. $\square$

Proposition 2.4 — *Let ρ be an integrable representation on (E, J'). The bilinear map δ given by*

$$\delta : \mathfrak{g}^{0,1} \times E^{1,0} \xrightarrow{\rho} E_{\mathbb{C}} \xrightarrow{pr} E^{1,0} \quad (\bar{X}, V) \mapsto \left(\rho(\bar{X})V\right)^{1,0}$$

induces a complex linear representation of $\mathfrak{g}^{0,1}$ on $E^{1,0}$.

Proof. The map is clearly complex bilinear and it remains to prove the compatibility with the bracket. Let $\bar{X}, \bar{Y} \in \mathfrak{g}^{0,1}$ and $V \in E^{1,0}$ be arbitrary. Note that $\rho(\bar{Y})V = \delta(\bar{Y}, V) + \left(\rho(\bar{Y})V\right)^{0,1}$. Then

$$\begin{aligned}
&\delta([\bar{X}, \bar{Y}], V)\\
=&(\rho([\bar{X}, \bar{Y}])(V))^{1,0}\\
=&\left(\rho(\bar{X})\rho(\bar{Y})V - \rho(\bar{Y})\rho(\bar{X})V\right)^{1,0}\\
=&\Big(\rho(\bar{X})(\delta(\bar{Y}, V) + \left(\rho(\bar{Y})V\right)^{0,1}) - \rho(\bar{Y})(\delta(\bar{X}, V) + \left(\rho(\bar{X})V\right)^{0,1})\Big)^{1,0}\\
=&\left(\rho(\bar{X})\delta(\bar{Y}, V) - \rho(\bar{Y})\delta(\bar{X}, V)\right)^{1,0} + (\underbrace{\rho(\bar{X})(\rho(\bar{Y})V)^{0,1} - \rho(\bar{Y})(\rho(\bar{X})V)^{0,1}}_{\text{of type (0,1)}})^{1,0}\\
=&\delta(\bar{X}, \delta(\bar{Y}, V)) - \delta(\bar{Y}, \delta(\bar{X}, V)).
\end{aligned}$$

Here we used that the action of $\mathfrak{g}^{0,1}$ maps $E^{0,1}$ to $E^{0,1}$ which is the complex conjugate of Proposition 2.2 (iii). Hence δ induces a $\mathfrak{g}^{0,1}$-module structure on $E^{1,0}$ as claimed. $\square$

Lemma 2.5 — *Let (E, J') be an integrable $(\mathfrak{g}, J)$-module. Then the dual module with the induced $\mathfrak{g}$-module structure is antiintegrable.*

Proof. If $x \in \mathfrak{g}$ and $\phi \in E^*$ then the induced module structure is given by $(x \cdot \phi)(v) = -\phi(xv)$ for $v \in E$. We have to show that for $\bar{X} \in \mathfrak{g}^{0,1}$ and $\Phi \in E^{*1,0}$ the map $(\bar{X} \cdot \Phi)$ annihilates $E^{0,1}$. But if $\bar{V}$ is in $E^{0,1}$ then by the above proposition $\bar{X}\bar{V} \in E^{0,1}$ and $\Phi(\bar{X}\bar{V}) = 0$. $\square$

Remark 2.6 — The above result seems unnatural only at first sight. If we consider E as left $\mathfrak{gl}(E)$-module in the canonical way then the complex structure $J \in \operatorname{End} E$ acts on the left. The dual vector space E^* comes with a natural action of $\mathfrak{gl}(E)$ on the right:

$$\phi \cdot A := \phi \circ A \qquad \text{for } A \in \mathfrak{gl}(E), \phi \in E^*$$

and the complex structure of E^* is given exactly in this way $J'^*\phi = \phi \circ J'$.

In order to make E^* a left $\mathfrak{gl}(E)$-module we have to change sign

$$A \cdot \phi := -\phi \circ A$$

which with regard to the complex structure corresponds to complex conjugation.

In fact, integrable modules do not behave well under standard linear algebra operation like the tensor product. The reason is simply that we have to work over $\mathbb{C}$ if we want to keep the complex structure on our vector spaces and over $\mathbb{R}$ if we want to preserve the $\mathfrak{g}$ action, since this action is not $\mathbb{C}$ linear in general.

2.2 Integrable modules and vector bundles

Now we want to relate the notion of integrable $\mathfrak{g}$-module to geometry. First we forget the complex structures and look at the differentiable situation:

Let $\mathfrak{g}$ be a real Lie algebra and G the corresponding simply connected Lie group. Let E be a (left) $\mathfrak{g}$-module and $\Gamma \subset G$ a co-compact discrete subgroup. Then $E \times G$ is the trivial bundle with an action of G on the left, given by the representation of $\mathfrak{g}$ on E.

If we take the quotient by the action of the subgroup Γ then the result is a homogeneous, flat vector bundle on $M = \Gamma\backslash G$.

Another possibility to look at this situation is the following: the representation of $\mathfrak{g}$ on E gives rise to a central extension, the semi-direct product $E \rtimes \mathfrak{g}$. The vector space underlying this Lie algebra is $E \oplus \mathfrak{g}$ with the Lie algebra structure given by $[(v,x),(w,y)] = (x \cdot w - y \cdot v, [x,y])$ for $(v,x),(w,y) \in E \oplus \mathfrak{g}$.

Regarding the real vector space E as a commutative Lie group, we get the exact sequence of Lie groups

$$0 \to E \to E \rtimes G \to G \to 1.$$

Now we take the complex structures $(\mathfrak{g}, J)$ and (E, J') into account: the complex structure on $\mathfrak{g}$ induces a left-invariant almost complex structure on G by declaring $(T_gG)^{1,0} := l_{g*}\mathfrak{g}^{1,0}$. Here we identify $\mathfrak{g}$ with the tangent space at the identity and l_g is the left multiplication by an element $g \in G$.

This almost complex structure is integrable if and only if J is integrable in the sense of (1)and we assume this to be the case in what follows. Hence G carries the structure of a complex manifold.

By definition left multiplication l_g is holomorphic but in general the multiplication on the right will not be holomorphic. If this is indeed the case, then J is bi-invariant and G is in fact a complex Lie group.

If we take the quotient by the action of the lattice on the left the result is a complex manifold on which the normaliser of Γ acts holomorphically on the left and the whole group acts differentiably on the right

Now consider a $(\mathfrak{g}, J)$-module with a complex structure (E, J'). We want to define the structure of a complex vector bundle on the differentiably trivial vector bundle $E \times G$ but since we want to be able to pass to the quotient $M = \Gamma\backslash G$ it should be compatible with the action on the left, i.e., left-invariant.

Lemma 2.7 — *The $(\mathfrak{g}, J)$-module (E, J') is integrable if and only if $J' \times J$ induces a left-invariant complex structure on the Lie group $E \rtimes G$.*

Proof. As remarked above we have to calculate the Nijenhuis tensor (1) for $I = J' \times J$ on the Lie algebra $E \rtimes \mathfrak{g}$. Let $(v, x), (w, y)$ be in $E \rtimes \mathfrak{g}$.

$$\begin{aligned}
&[(v,x),(w,y)] - [I(v,x), I(w,y)] + I[I(v,x),(w,y)] + I[(v,x), I(w,y)]\\
=&(x\cdot w - y\cdot v, [x,y]) - (Jx\cdot J'w - Jy\cdot J'v, [Jx, Jy])\\
&+ (J'(Jx\cdot w - y\cdot J'v), J[Jx,y]) + (J'(x\cdot J'w - Jy\cdot v), J[x, Jy])
\end{aligned}$$

The second component yields the Nijenhuis tensor for J and hence vanishes since we assumed J to be integrable. We calculate the first component using the notation $\rho(x)$ for the element in $\operatorname{End} E$ corresponding to the action of x on E.

$$\begin{aligned}
&x\cdot w - y\cdot v - Jx\cdot J'w + Jy\cdot J'v\\
&\qquad + J'(Jx\cdot w - y\cdot J'v) + (J'(x\cdot J'w - Jy\cdot v)\\
=&(\rho(x) - \rho(Jx)\circ J + J'\circ\rho(Jx) + J'\circ\rho(x)\circ J')w\\
&\qquad - (\rho(y) - \rho(Jy)\circ J + J'\circ\rho(Jy) + J'\circ\rho(y)\circ J')v\\
=:&\Phi(x)w - \Phi(y)v
\end{aligned}$$

Furthermore

$$\begin{aligned}
\Phi(x)w =&(\rho(x) - \rho(Jx)\circ J + J'\circ\rho(Jx) + J'\circ\rho(x)\circ J')w\\
=&(-J'^2\rho(x) + J'\circ\rho(x)\circ J' + [J', \rho(Jx)])w\\
=&(J'[\rho(x), J'] + [J', \rho(Jx)])w\\
=&\mathcal{N}(x)w
\end{aligned}$$

and hence the second component is equal to $\mathcal{N}(x)w - \mathcal{N}(y)v$. Setting $v = 0$ we see that I is integrable if and only if (E, J') is an integrable $(\mathfrak{g}, J)$-module. □

Remark 2.8 — (*i*) Note that the left invariance of a complex structure is not preserved by standard vector bundle operations for essentially the same reason as explained in remark 2.6.

(*ii*) Even if a vector bundle E is equipped with an integrable left-invariant structure left-invariant sections are not necessarily holomorphic. In fact, the action of each $g \in G$ on E is holomorphic but there is no natural way to speak about *the action varying holomorphically* since G is not a complex Lie group in general.

2.3 Lie algebra Dolbeault cohomology

In this section we want to define a cohomology theory for Lie algebras with a complex structure with values in a finite dimensional integrable module. In the notation we will often suppress the complex structures.

Recall that the cohomology groups of a Lie algebra $\mathfrak{g}$ with values in a $\mathfrak{g}$-module E are defined as the right derived functors of the functor of invariants ([Wei94], chap. 7)

$$E \mapsto E^{\mathfrak{g}} = \{m \in E \mid x \cdot m = m \text{ for all } x \in \mathfrak{g}\}.$$

The cohomology groups can be calculated by taking the cohomology of the following complex, called Chevalley complex for E:

$$0 \to E \xrightarrow{d_0} \mathfrak{g}^* \otimes E \xrightarrow{d_1} \Lambda^2\mathfrak{g}^* \otimes E \xrightarrow{d_2} \ldots \to \Lambda^{dim\mathfrak{g}}\mathfrak{g}^* \otimes E \to 0$$

with differential given by

$$\begin{aligned}(d_k\alpha)(x_1, \ldots, x_{k+1}) := &\sum_{i=1}^{k+1}(-1)^{i+1}x_i(\alpha(x_1, \ldots, \hat{x}_i, \ldots, x_{k+1})) \\ &+ \sum_{1\leq i<j\leq k+1} (-1)^{i+j}\alpha([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1}). \quad (2)\end{aligned}$$

This is the complex originally introduced by Chevalley and Eilenberg [Kna02, CE48]. We will use it because it is finite dimensional and has a nice interpretation in the geometric context (see Section 2.4).

Definition 2.9 — *Let $(\mathfrak{g}, J)$ be a Lie algebra with complex structure and let (E, J') be a finite dimensional , integrable (antiintegrable) $\mathfrak{g}$-module. Then we define*

$$H^{p,q}(\mathfrak{g}, E) = H^{p,q}_{\overline{\partial}}((\mathfrak{g}, J), (E, J')) := H^q(\mathfrak{g}^{0,1}, \Lambda^p\mathfrak{g}^{1,0} \otimes E^{1,0})$$

*where $H^k(\mathfrak{g}^{0,1}, E^{1,0})$ is the Lie algebra cohomology of $\mathfrak{g}^{0,1}$ with values in the $E^{1,0}$ with the module structure as in Proposition 2.4 (as in Corollary 2.3). We call $H^k_{\overline{\partial}}(\mathfrak{g}, E) := H^{0,k}(\mathfrak{g}, E)$ the k-th **Dolbeault cohomology group of** $\mathfrak{g}$ **with values in** E.*

Example 2.10 — Consider for $(\mathfrak{g}, J)$ as above $\mathbb{C}$ as the trivial $\mathfrak{g}_{\mathbb{C}}$-module. Then the associated Chevalley differential on the exterior algebra $\Lambda^\bullet\mathfrak{g}_{\mathbb{C}}$ decomposes into $d = \partial + \overline{\partial}$ since J is integrable and we can consider the double complex $(\Lambda^{p,q}\mathfrak{g}^*, \overline{\partial}, \partial)$.

The adjoint action of $\mathfrak{g}$ on itself yields an antiintegrable $\mathfrak{g}$-module structure on $\mathfrak{g}^*$. Hence we have a $\mathfrak{g}^{0,1}$-module structure on $\Lambda^p\mathfrak{g}^{1,0}$. It is now easy to see that the columns of the above double complex

$$0 \to \Lambda^{p,0}\mathfrak{g} \to \Lambda^{p,1}\mathfrak{g} \to \ldots$$

are in fact the Chevalley complexes calculating $H^q(\mathfrak{g}^{0,1}, \Lambda^{p,0}\mathfrak{g}) = H^{p,q}_{\overline{\partial}}(\mathfrak{g}, \mathbb{C})$.

Now we want to develop some kind of Hodge theory for our Dolbeault cohomology which we model on the usual Hodge theory for holomorphic vector bundles as it can be found for example in the book of Huybrechts [Huy05]. Let $2n$ be the real dimension of $\mathfrak{g}$.

First of all we choose an euclidean structure $g = \langle -, - \rangle$ on the real vector space underlying $\mathfrak{g}$ which is compatible with the given complex structure J in the sense that $\langle J-, J-\rangle = \langle -, -\rangle$. Let vol be the volume form, i.e., the unique element in $\Lambda^{2n}\mathfrak{g}^*$ inducing the same orientation as J and of length one in the induced metric on $\Lambda^\bullet\mathfrak{g}^*$. We have the Hodge $*$-operator given by the defining relation

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle vol \quad \text{for all } \alpha, \beta \in \Lambda^\bullet\mathfrak{g}^*$$

which is an isometry on $\Lambda^\bullet\mathfrak{g}^*$. On the complexified vector space $\mathfrak{g}_\mathbb{C}$ we have a natural, induced hermitian product $\langle -, - \rangle_\mathbb{C}$ and the $*$-operator determined by

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle_\mathbb{C} vol \quad \text{for all } \alpha, \beta \in \Lambda^\bullet\mathfrak{g}^*_\mathbb{C}.$$

which maps (p, q)-forms to $(n-p, n-q)$-forms.

We want now to define a star operator also on $\Lambda^{\bullet,\bullet}\mathfrak{g}^* \otimes E^{1,0}$. For this purpose we choose an euclidean product on E compatible with the complex structure J' which induces a hermitian structure h on $E^{1,0}$. We consider h as an $\mathbb{C}$-antilinear isomorphism $h : E^{1,0} \cong E^{*1,0}$. Then

$$\bar{*}_E : \Lambda^{p,q}\mathfrak{g}^* \otimes E^{1,0} \to \Lambda^{n-p,n-q} \otimes E^{1,0*}$$

is defined by $\bar{*}_E(\alpha \otimes s) = \overline{*\alpha} \otimes h(s) = *(\bar{\alpha}) \otimes h(s)$. Let $(-, -)$ be the hermitian form on $\Lambda^{\bullet,\bullet}\mathfrak{g}^* \otimes E^{1,0}$ induced by g and h. Then $\bar{*}_E$ is a $\mathbb{C}$-antilinear isomorphism depending on our choice of g and h and the identity

$$(\alpha, \beta) vol = \alpha \wedge \bar{*}_E\beta$$

holds for $\alpha, \beta \in \Lambda^{p,q}\mathfrak{g} \otimes E^{1,0}$, where " $\wedge$ " is the exterior product for the elements in $\Lambda^{\bullet,\bullet}\mathfrak{g}$ and the evaluation map $E^{1,0} \otimes E^{1,0*} \to \mathbb{C}$ on the module part. It is not difficult to verify that one has $\bar{*}_{E^*} \circ \bar{*}_E = (-1)^{p+q}$ on $\Lambda^{p,q}\mathfrak{g} \otimes E^{1,0}$.

Definition 2.11 — *Let (E, J') be an (anti-) integrable $(\mathfrak{g}, J)$-module. The operator $\overline{\partial}_E^* : \Lambda^{p,q}\mathfrak{g} \otimes E^{1,0} \to \Lambda^{p,q-1}\mathfrak{g} \otimes E^{1,0}$ is defined as*

$$\overline{\partial}_E^* := -\bar{*}_{E^*} \circ \overline{\partial}_{E^*} \circ \bar{*}_E.$$

Let $\Delta_E := \overline{\partial}_E^*\overline{\partial}_E + \overline{\partial}_E\overline{\partial}_E^*$ be the ***Laplace operator*** *on $\Lambda^{p,q}\mathfrak{g} \otimes E^{1,0}$. We call an element α harmonic if $\Delta_E(\alpha) = 0$ and denote by $\mathcal{H}^{p,q}(\mathfrak{g}, E)$ the space of harmonic elements (where we omit g and h from the notation).*

Observe that $\bar{*}_E$ induces a $\mathbb{C}$-antilinear isomorphism

$$\bar{*}_E : \mathcal{H}^{p,q}(\mathfrak{g}, E) \cong \mathcal{H}^{n-p,n-q}(\mathfrak{g}, E^*).$$

Proposition 2.12 — *In the above situation the operator $\overline{\partial}_E^*$ is adjoint to $\overline{\partial}_E$ with respect to the metric induced by g and h if $H^{2n}(\mathfrak{g}_{\mathbb{C}}, \mathbb{C}) = \mathbb{C}$, where $\mathbb{C}$ is considered as the trivial $\mathfrak{g}$-module. In this case Δ_E is selfadjoint.*

The condition on the cohomology is somehow the equivalent of Stokes theorem as will be seen in the proof.

Proof. The second assertion is a consequence of the first one which in turn is proved by the following calculation:

First of all note that the assumption $\Lambda^{2n}\mathfrak{g}_{\mathbb{C}}^* \cong \mathbb{C}$ implies that $d_{2n-1} = 0$ in $\Lambda^\bullet\mathfrak{g}^*$, the Chevalley complex of the trivial module. Hence the same holds for $\overline{\partial} : \Lambda^{n,n-1}\mathfrak{g}^* \to \Lambda^{n,n}\mathfrak{g}^*$. For $\alpha \in \Lambda^{p,q}\mathfrak{g}^* \otimes E^{1,0}$ and $\beta \in \Lambda^{p,q+1}\mathfrak{g}^* \otimes E^{1,0}$ we have

$$\begin{aligned}
(\alpha, \overline{\partial}_E^*\beta)vol &= -(\alpha, \bar{*}_{E^*} \circ \overline{\partial}_{E^*} \circ \bar{*}_E\beta)vol \\
&= -\alpha \wedge \bar{*}_E\bar{*}_{E^*}\overline{\partial}_{E^*}\bar{*}_E\beta \\
&= (-1)^{n-p+n-q-1}\alpha \wedge \overline{\partial}_{E^*}\bar{*}_E\beta \\
&= -\overline{\partial}(\alpha \wedge \bar{*}_E\beta) + \overline{\partial}_E\alpha \wedge \bar{*}_E\beta \\
&= \overline{\partial}_E\alpha \wedge \bar{*}_E\beta \\
&= (\overline{\partial}_E\alpha, \beta)vol.
\end{aligned}$$

Here we used the identity

$$\overline{\partial}(\alpha \wedge \bar{*}_E\beta) = \overline{\partial}_E\alpha \wedge \bar{*}_E\beta + (-1)^{p+q}\alpha \wedge \overline{\partial}_{E^*}\bar{*}_E\beta$$

that follows form the Leibniz rule in the exterior algebra and the fact that the evaluation map $E^{1,0} \otimes E^{1,0^*} \to \mathbb{C}$ is a map of $\mathfrak{g}^{0,1}$-modules. □

Remark 2.13 — We have always $H^{2n}(\mathfrak{g}_{\mathbb{C}}, \Lambda^{2n}\mathfrak{g}_{\mathbb{C}}) = \mathbb{C}$ (See [Wei94], Exercise 7.7.2). Hence the assumptions of the theorem hold if $\mathfrak{g}$ acts trivially on $\Lambda^{2n}\mathfrak{g}$, in particular if $\mathfrak{g}$ is nilpotent.

Here are some standard consequences:

Corollary 2.14 — *If $H^{2n}(\mathfrak{g}_{\mathbb{C}}, \mathbb{C}) = \mathbb{C}$ then an element $\alpha \in \Lambda^{p,q}\mathfrak{g} \otimes E$ is harmonic if and only if α is $\overline{\partial}_E$ and $\overline{\partial}_E^*$ closed.*

Proof. Standard argument. □

Corollary 2.15 (Hodge decomposition) — *Let (E, J) be a (anti-)integrable module over the Lie algebra with complex structure $(\mathfrak{g}, J)$ both equipped with a compatible euclidean product. If $H^{2n}(\mathfrak{g}_{\mathbb{C}}, \mathbb{C}) = \mathbb{C}$ then there is a orthogonal decomposition*

$$\Lambda^{p,q} \otimes E^{1,0} = \overline{\partial}_E(\Lambda^{p,q-1} \otimes E^{1,0}) \oplus \mathcal{H}^{p,q}(\mathfrak{g}, E) \oplus \overline{\partial}_E^*(\Lambda^{p,q+1} \otimes E^{1,0}).$$

Proof. Since everthing is finite dimensional this follows trivially from the above. □

Corollary 2.16 — *If $H^{2n}(\mathfrak{g}_{\mathbb{C}}, \mathbb{C}) = \mathbb{C}$ then the natural projection*

$$\mathcal{H}^{p,q}(\mathfrak{g}, E) \to H^{p,q}(\mathfrak{g}, E)$$

is bijective.

Theorem 2.17 (Serre Duality) — *Let $(\mathfrak{g}, J)$ be a Lie algebra with complex structure such that $H^{2n}(\mathfrak{g}_{\mathbb{C}}, \mathbb{C}) = \mathbb{C}$ and (E, J') an (anti-)integrable $\mathfrak{g}$-module. Then the paring*

$$H^{p,q}(\mathfrak{g}, E) \times H^{n-p,n-q}(\mathfrak{g}, E^*) \to \mathbb{C} \cdot vol \cong \mathbb{C} \quad (\alpha, \beta) \mapsto \alpha \wedge \beta$$

is well defined and non degenerate where " $\wedge$ " is as above.

Proof. Fix hermitian structures on E and $\mathfrak{g}$ respectively. Then consider the pairing

$$\mathcal{H}^{p,q}(\mathfrak{g}, E) \times \mathcal{H}^{n-p,n-q}(\mathfrak{g}, E^*) \to \mathbb{C} \cdot vol \cong \mathbb{C}.$$

We claim that for any non zero $\alpha \in \mathcal{H}^{p,q}(\mathfrak{g}, E)$ there exists an element $\beta \in \mathcal{H}^{n-p,n-q}(\mathfrak{g}, E^*)$ such that $\alpha \wedge \beta \neq 0$. Indeed, choosing $\beta = \bar{*}_E \alpha$ we have

$$\alpha \wedge \beta = \alpha \wedge \bar{*}_E \alpha = (\alpha, \alpha) vol = \|\alpha\|^2 vol \neq 0.$$

This proves that the pairing in non degenerate. □

Corollary 2.18 — *Let $(\mathfrak{g}, J)$ be a Lie algebra with complex structure such that $H^{2n}(\mathfrak{g}_{\mathbb{C}}, \mathbb{C}) = \mathbb{C}$ For any (anti-)integrable $(\mathfrak{g}, J)$-module there exist natural isomorphisms*

$$H^{p,q}(\mathfrak{g}, E) \cong H^{n-p,n-q}(\mathfrak{g}, E^*)^*$$

and if $\Lambda^n \mathfrak{g}^*$ is the trivial $\mathfrak{g}$-module

$$H^q_{\bar{\partial}}(\mathfrak{g}, E) \cong H^{n-q}_{\bar{\partial}}(\mathfrak{g}, E^*)^*.$$

2.4 Cohomology with invariant forms

We are now going to translate our results on Lie algebra Dolbeault cohomology to the geometric situation.

Recall the situation considered in Section 2.2: Let $(\mathfrak{g}, J)$ be a real Lie algebra with complex structure of real dimension $2n$ and (E, J') an integrable $(\mathfrak{g}, J)$-module. Let G be the simply connected Lie group associated to $\mathfrak{g}$ endowed with the left-invariant complex structure induced by J. Let Γ a uniform lattice in G and consider the flat, homogeneous, holomorphic vector bundle $\mathcal{E}$ on $M = \Gamma\backslash G$ constructed by taking the quotient of $E \times G$ by Γ acting on the left.

Let g be an euclidean structure on $\mathfrak{g}$ compatible with the complex structure J, such that M has volume one with respect to the associated left-invariant metric on M. Choose also an euclidean structure on E compatible with the complex structure J'.

Let $\pi : G \to M$ be the projection. We say that a smooth section $s \in \mathcal{A}^{p,q}(M, \mathcal{E})$ is invariant if $\pi^* s$ is invariant under the action of G on the left, i.e., $l_g^*(\pi^* s) = \pi^* s$ for all $g \in G$. This makes sense since $\pi^*\mathcal{E} = E \times G$ is trivial as a smooth vector bundle and in particular $l_g^* \pi^* \mathcal{E} = \pi^* \mathcal{E}$. Note that a smooth section s in the trivial bundle $E \times G$ is the pullback of a section of $\mathcal{E}$ if and only if it is invariant under the action of Γ on the left.

The relation between the usual Dolbeault theory for vector bundles on complex manifolds and our theory developed so far is summarised in the following

Proposition 2.19 — *In this situation $\Lambda^{p,q}\mathfrak{g} \otimes E^{1,0}$ can be identified with the subset of invariant, smooth differential form on M with values in the holomorphic bundle $\mathcal{E}$. Assume further that $H^{2n}(\mathfrak{g}_{\mathbb{C}}, \mathbb{C}) = \mathbb{C}$. Then the following holds:*

- (*i*) *The differential in the Chevalley complex as given in* (2) *coincides with the usual differential restricted to invariant forms with values in $\mathcal{E}$. In particular if E is the trivial module the decomposition $d = \partial + \overline{\partial}$ on $\Lambda^{\bullet,\bullet}\mathfrak{g}^*$ coincides with the usual one on the complex manifold M.*

- (*ii*) *The Chevalley complex associated to the $\mathfrak{g}^{0,1}$-module structure on $E^{1,0}$ is the subcomplex of invariant forms contained in the usual Dolbeault resolution of the holomorphic vector bundle $\mathcal{E}$ by smooth differential forms with values in $\mathcal{E}$.*

- (*iii*) *The Hodge $*$-operator defined on $\Lambda^{\bullet}\mathfrak{g}_{\mathbb{C}}^*$ in Section 2.3 coincides with the usual Hodge $*$-operator on the exterior algebra of smooth differential forms. The same holds true for the operator $\bar{*}_E$.*

- (*iv*) *The operators $\overline{\partial}_E^*$ and Δ_E in definition 2.11 are the restrictions of the corresponding operators on smooth differential forms. In particular we have an inclusion*

$$\mathcal{H}^{n-p,n-q}(\mathfrak{g}, E) \subset \mathcal{H}^{n-p,n-q}(M, \mathcal{E})$$

  where $\mathcal{H}^{n-p,n-q}(M, \mathcal{E})$ are the harmonic (p, q) forms with values in $\mathcal{E}$ with respect to the invariant hermitian structures chosen above.

Proof. The first claim is is clear by construction. The Lie bracket on $\mathfrak{g}$ is clearly the restriction of the usual Lie bracket on vector fields on M

and also the definition of the differential in (2) coincides with the usual one for smooth differential forms (see e.g. [Huy05], p. 283). Since $\pi^*\mathcal{E}$ is differentiably a trivial bundle the same holds for differential forms on G with values in $\pi^*\mathcal{E}$ and therefore also for sections of $\mathcal{E}$ itself since we can check this locally. This proves (i) and (ii) using the identification of $\mathcal{E}$ with $E^{1,0}$.

Our reference for the Hodge theory of holomorphic vector bundles is [Huy05] (ch. 4.1). Now, recall that we defined our operator $\bar{*}_E$ by the relation

$$\alpha \wedge \bar{*}_E\beta = (\alpha, \beta)vol = (\alpha, \beta) * 1$$

for $\alpha, \beta \in \Lambda^{p,q}\mathfrak{g} \otimes E^{1,0}$ which conicides with the definition for differential forms in $\mathcal{A}^{p,q}(M, \mathcal{E})$ if we consider α and β as invariant differential forms on M:

The hermitian metric on $\mathcal{A}^{p,q}(M, \mathcal{E})$ is defined by $(\alpha, \beta)_M = \int_M (\alpha, \beta)vol$ but if the forms are invariant we have

$$(\alpha, \beta)_M = \int_M (\alpha, \beta)vol = (\alpha, \beta) \int_M vol = (\alpha, \beta)$$

since chose the invariant metric such that the volume of M is one.

Therefore also $\bar{*}_{\mathcal{E}} = \bar{*}_E$ and this concludes the proof since the Laplace operator can be described in terms of $\bar{*}_{\mathcal{E}}$ and $\overline{\partial}$. □

Corollary 2.20 — *In the above situation we have an inclusion*

$$\iota_E : H^{p,q}(\mathfrak{g}, E) \to H^{p,q}(M, \mathcal{E})$$

induced by the inclusion on the level of harmonic differential forms. In particular if ι_E is an isomorphism then so is $\iota_{E^*} : H^{n-p,n-q}(\mathfrak{g}, E^*) \to H^{n-p,n-q}(M, \mathcal{E}^*)$.

Proof. The first claim is an immediate consequence of (vi) in the proposition while the second then follows for dimension reasons from Serre-Duality both on M and for Lie algebra Dolbeault cohomology (Corollary 2.18). □

We will apply this to the cohomology of nilmanifolds in the next section in order to study the space of infinitesimal deformations.

3 Dolbeault cohomology of nilmanifolds and small deformations

The aim of this section is to prove that in the generic case small deformations of nilmanifolds with left-invariant complex structure carry a left-invariant complex structure.

We want to parametrise the space of left-invariant complex structures on a given nilpotent Lie algebra $\mathfrak{g}$. Let $2n$ be the real dimension of $\mathfrak{g}$. We continue to use the notations of Section 1.

A complex structure J is uniquely determined by specifying the $(0,1)$-subspace $\bar{V} \subset \mathfrak{g}_\mathbb{C}$ and the integrability condition can be expressed as $[\bar{V}, \bar{V}] \subset \bar{V}$. Hence we write (like in [Sal01])

$$\mathcal{C}(\mathfrak{g}) := \{\bar{V} \in \mathbb{G}(n, \mathfrak{g}_\mathbb{C}) \mid V \cap \bar{V} = 0, [\bar{V}, \bar{V}] \subset \bar{V}\}$$

where $\mathbb{G}(n, \mathfrak{g}_\mathbb{C})$ is the Grassmannian of n-dimensional complex subspaces of $\mathfrak{g}_\mathbb{C}$.

Recall that the tangent space at a point $\bar{V}$ is

$$T_{\bar{V}}\mathbb{G}(n, \mathfrak{g}_\mathbb{C}) = \mathrm{Hom}_\mathbb{C}(\bar{V}, \mathfrak{g}_\mathbb{C}/\bar{V}) \cong \bar{V}^* \otimes V \cong \mathfrak{g}^{0,1^*} \otimes \mathfrak{g}^{1,0}$$

if we endow $\mathfrak{g}$ with the complex structure $J_{\bar{V}}$ induced by $\bar{V}$.

In general it is a difficult question to decide if $\mathcal{C}(\mathfrak{g})$ is non empty for a given Lie algebra $\mathfrak{g}$. For the next paragraph we will assume this to be the case.

Now fix a simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$. We want to describe a family of complex manifolds $\pi : \mathcal{M}(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g})$ such that over every point $\bar{V} \in \mathcal{C}(\mathfrak{g})$ the fibre $\pi^{-1}(\bar{V})$ is the manifold G with the left-invariant complex structure $J_{\bar{V}}$.

Let $\bar{\mathcal{V}} \subset \mathfrak{g}_\mathbb{C} \times \tilde{\mathcal{C}}(\mathfrak{g})$ be the restriction of the tautological bundle on the Grassmannian to the open subset

$$\tilde{\mathcal{C}}(\mathfrak{g}) := \{\bar{V} \in \mathbb{G}(n, \mathfrak{g}_\mathbb{C}) \mid V \cap \bar{V} = 0\}$$

and consider the manifold

$$\tilde{\mathcal{M}}(\mathfrak{g}) := G \times \tilde{\mathcal{C}}(\mathfrak{g}).$$

The group G acts on on the left of $\tilde{\mathcal{M}}(\mathfrak{g})$ by $l_g(h, \bar{V}) = (gh, \bar{V})$ and we can define the subbundle $T^{0,1}\tilde{\mathcal{M}}(\mathfrak{g}) \subset T\tilde{\mathcal{M}}(\mathfrak{g})_\mathbb{C}$ by

$$T^{0,1}\tilde{\mathcal{M}}(\mathfrak{g})\big|_{\{g\}\times\tilde{\mathcal{C}}(\mathfrak{g})} := l_{g*}\bar{\mathcal{V}} \oplus T^{0,1}\tilde{\mathcal{C}}(\mathfrak{g}).$$

This subbundle gives an almost complex structure on $\tilde{\mathcal{M}}(\mathfrak{g})$ which is integrable over $\mathcal{C}(\mathfrak{g})$. So we obtain our desired family by taking the pullback

$$\mathcal{M}(\mathfrak{g}) := \tilde{\mathcal{M}}(\mathfrak{g}) \times_{\tilde{\mathcal{C}}(\mathfrak{g})} \mathcal{C}(\mathfrak{g}).$$

If $\Gamma \subset G$ is a lattice then we can take the quotient of $\mathcal{M}(\mathfrak{g})$ by the action of Γ on the left and we obtain a family $\mathcal{M}(\mathfrak{g}, \Gamma) \to \mathcal{C}(\mathfrak{g})$ of compact, complex manifolds such that the fibre over $\bar{V} \in \mathcal{C}(\mathfrak{g})$ is the nilmanifold $M_{\bar{V}} = (\mathfrak{g}, \Gamma, J_{\bar{V}})$. Summarising we have shown the following:

Proposition 3.1 — *Every nilmanifold $M' = (\mathfrak{g}', J', \Gamma')$ with fundamental group $\Gamma' \cong \Gamma$ is isomorphic to a nilmanifold in the family $\mathcal{M}(\mathfrak{g}, \Gamma)$.*

Proof. We only have to observe that by [VGS00], p. 45, corollary 2.8 the lattice Γ' determines G' and $\mathfrak{g}'$ up to canonical isomorphism hence M' isomorphic to a fibre in the family $\mathcal{M}(\mathfrak{g}, \Gamma) \to \mathcal{C}(\mathfrak{g})$. ☐

Remark 3.2 — There are many natural questions concerning the family $\mathcal{C}(\mathfrak{g})$, for example when is it non-empty, smooth, versal and what are the connected components. Catanese and Frediani studied in [Cat04, CF06] the subfamily consisting of principal holomorphic torus bundles over a torus with fixed dimension of fibre and base, the so called *Appel-Humbert family*, and proved that in some 3-dimensional cases it is a connected component of the Teichmüller-Space.

We will now use deformation theory in the spirit of Kodaira-Spencer and Kuranishi to study small deformations of nilmanifolds. In order to do this we need a good description of the cohomology of the tangent bundle.

By a theorem of Nomizu [Nom54] the de Rham cohomology of a nilmanifold can be calculated using invariant differential forms and is isomorphic to the cohomology of the complex

$$0 \to \mathfrak{g}^* \xrightarrow{d} \Lambda^2\mathfrak{g}^* \xrightarrow{d} \Lambda^3\mathfrak{g}^* \xrightarrow{d} \dots$$

The question if the Dolbeault cohomology of compact nilmanifolds with left-invariant complex structure can be calculated using invariant differential forms has been addressed by Console and Fino in [CF01] and Cordero, Fernandez, Gray and Ugarte in [CFGU00]. We restate their results using our notation from Section 1.2 and 2.3:

Theorem 3.3 — *Let $\Gamma \backslash G = M$ be a real nilmanifold with Lie algebra $\mathfrak{g}$. Then there is a dense open subset U of the space $\mathcal{C}(\mathfrak{g})$ of all left-invariant complex structures on M such that for all $J \in U$ we have an isomorphism*

$$\iota_J : H^{p,q}((\mathfrak{g}, J), \mathbb{C}) \to H^{p,q}(M_J),$$

on the associated nilmanifold with left-invariant complex structure $M_J = (\mathfrak{g}, J, \Gamma)$, where we consider $\mathbb{C}$ as the trivial $\mathfrak{g}_{\mathbb{C}}$-module ([CF01], Theorem A). In particular this holds true in the following cases:

- *The complex structure J is Γ-rational. ([CF01], Theorem B)*

- *The complex structure J is abelian [CF01].*
- *The complex structure J is bi-invariant, G is a complex Lie group and M_J is complex parallelisable [Sak76, CF01].*
- *The complex manifold M_J has the structure of an iterated principal holomorphic torus bundle, i.e., the complex structure is nilpotent and there exists a torus bundles series $(\mathcal{S}^i\mathfrak{g})_{i=1,\dots,t}$ in $\mathfrak{g}$ such that the subspaces $\mathcal{S}^i\mathfrak{g}$ are rational with respect to the lattice Γ (see Section 1.2 for the definitions) [CFGU00].*

  This is always the case if $\mathfrak{g}$ admits a stable torus bundle series (Definition 1.10).

The idea of the proof is the following: as long as M_J can be given a structure of iterated bundle with a good control over the cohomology of the base and of the fibre one can use the Borel spectral sequence for Dolbeault cohomology in order to get an inductive proof. This is the case if the complex structure is Γ-rational or M_J is an iterated principal holomorphic bundle. This yields the result on a dense subset of the space of invariant complex structures and Console and Fino then show that the property *"The map ι_J is an isomorphism."* is stable under small deformations.

It is an open question if ι_J is in fact an isomorphism for every left-invariant complex structure on a nilmanifold.

For a manifold M we denote by $\mathcal{T}_M$ the holomorphic tangent bundle and by $\Theta_M = \mathcal{O}(\mathcal{T}_M)$ the sheaf of holomorphic tangent vectors. Combining Theorem 3.3 with Corollary 2.20 we get:

Corollary 3.4 — *Under the same condition as in the theorem we have an isomorphism*

$$\iota : H^p_{\bar\partial}((\mathfrak{g}, J), \mathfrak{g}) \to H^p(M_J, \mathcal{T}_{M_J}) \cong H^p(M_J, \Theta_{M_J})$$

induced by the inclusion on the level of differential forms. Here we consider $\mathfrak{g}$ as an integrable $\mathfrak{g}$-module under the adjoint representation.

Proof. Just note that in the situation of the theorem

$$H^q_{\bar\partial}(\mathfrak{g}, \mathfrak{g}^*) = H^{1,q}((\mathfrak{g}, J), \mathbb{C}) \cong H^{1,q}(M_J) \cong H^q(M_J, \Omega^1_{M_J})$$

and using Serre duality on both sides we see that for dimensional reasons also ι is an isomorphism. □

The same was proved for 2-step nilmanifolds with abelian complex structure in [MPPS06] and for abelian complex structures in general in [CFP06]. Hence we can extend the theorem proved there to the general case:

Theorem 3.5 — *Let $M_J = (\mathfrak{g}, J, \Gamma)$ be a nilmanifold with left-invariant complex structure such that $\iota : H^{1,q}((\mathfrak{g}, J), \mathbb{C}) \to H^{1,q}(M_J)$ is an isomorphism for all q. Then all small deformations of the complex structure J are again left-invariant complex structures.*

By this we mean the following: Assume that we have a smooth family of compact, complex manifolds $\pi : \mathcal{M} \to \mathcal{B}$ such that the fibre over the point $0 \in \mathcal{B}$ is isomorphic to a nilmanifold with left-invariant complex structure $M = (\mathfrak{g}, J, \Gamma)$. Then there is an open neighbourhood U of 0 in $\mathcal{B}$ such that for every $t \in U$ the fibre over t is again a nilmanifold with left-invariant complex structure (with the same Lie algebra and lattice). Equivalently one could say that there is around J a small open subset of the space of integrable complex structures on $M = \Gamma\backslash G$ which contains only left-invariant complex structures.

Proof. By the work of Kuranishi the small deformations of M_J are governed by the differential graded algebra $\mathcal{A}^*_{M_J}(\mathcal{T}_M)$ of differential forms with values in $\mathcal{T}_M$. By the above corollary the inclusion $\Lambda^*\mathfrak{g}^{*0,1} \otimes \mathfrak{g}^{1,0} \subset \mathcal{A}^*_{M_J}(\mathcal{T}_M)$ is a quasi isomorphism and hence induces an isomorphism of corresponding deformation spaces.

We spell this out more in detail following Kuranishi's inductive method on harmonic forms in order to give a description of the Kuranishi space. Note that this has already been done in [MPPS06] in the context of abelian complex structures. We choose an invariant, compatible hermitian structure on M as in Section 2.4. Recall that the Shouten bracket is defined by

$$[\cdot,\cdot] : H^1(M, \mathcal{T}_M) \times H^1(M, \mathcal{T}_M) \to H^2(M, \mathcal{T}_M)$$
$$[\bar\omega \otimes V, \bar\omega' \otimes V] := \bar\omega' \wedge L_{V'}\bar\omega \otimes V + \bar\omega \wedge L_V\bar\omega' \otimes V' + \bar\omega \wedge \bar\omega' \otimes [V, V']$$

where L is the Lie derivative, i.e. $L_V\bar\omega' = i_V \circ d\bar\omega' + d \circ i_V\bar\omega'$. By assumption we can represent every class in $H^1(M, \mathcal{T}_M)$ by an element in $\mathcal{H}^{0,1}(\mathfrak{g}, \mathfrak{g})$ which can be considered as an invariant, harmonic differential form on M with respect to the hermitian structure.

Let G be Green's operator which inverts the Laplacian on the orthogonal complement of the harmonic forms. By construction G maps invariant forms to invariant forms since the Laplacian has this property. Let $\eta_1, \ldots, \eta_m$ be a basis for $\mathcal{H}^{0,1}(\mathfrak{g}, \mathfrak{g})$ and consider the equation

$$\phi(t) = \sum_{i=1}^{m} \eta_i t_i + \frac{1}{2}\overline{\partial}^* G[\phi(t), \phi(t)].$$

It has a formal power series solution with values in $\mathfrak{g}^{*0,1} \otimes \mathfrak{g}^{1,0}$ which is given inductively by

$$\phi_1(t) = \sum_{i=1}^{m} \eta_i t_i \text{ and } \phi_r(t) = \frac{1}{2}\sum_{s=1}^{r-1} \overline{\partial}^* G[\phi_s(t), \phi_{r-s}(t)].$$

Note that by construction $\phi(t)$ is left-invariant.

By Kuranishi theory (see e.g. [Cat88], p. 11) this series converges for small t and there is a complete family of deformations of M over the base

$$B := \{t \in B_\epsilon(0) \mid \overline{\partial}\phi(t) - \frac{1}{2}[\phi(t), \phi(t)] = 0\}.$$

If $\xi_1, \ldots, \xi_k$ is a basis of $\mathcal{H}^{0,2}(\mathfrak{g}, \mathfrak{g})$ then we can use the inner product $(\cdot, \cdot)$ on $\Lambda^2 \mathfrak{g}^{*0,1} \otimes \mathfrak{g}^{1,0}$ to describe B as the zero locus of the functions

$$g_i(t) = (\xi_i, [\phi(t), \phi(t)]), \qquad i = 1, \ldots, k.$$

The complex structure over a point $\eta = \sum_{i=1}^m \eta_i t_i \in B$ is determined by

$$(TM_\eta)^{0,1} = (id + \phi(t))TM^{0,1}.$$

In particular the complex structure is left-invariant since this is true for $\phi(t)$ and $TM^{0,1}$. □

We return to our special case: Going back to Example 1.13 recall that a principal holomorphic torus bundle over an elliptic curve M which is not a product admits a stable torus bundle series

$$\mathfrak{g} \supset \mathcal{Z}\mathfrak{g} \supset 0$$

on the corresponding Lie algebra. Hence the Dolbeault cohomology can be calculated using invariant forms for any complex structure on $\mathfrak{g}$. Using Theorem 3.5 this implies immediately:

Corollary 3.6 — *Let $f : M \to E$ be a principal holomorphic torus bundle over an elliptic curve which is not a product. Then every small deformation of M has the same structure.*

Generalising this, we reformulate the theorem in order to apply it to all elements in the family $\mathcal{C}(\mathfrak{g})$ at the same time.

Corollary 3.7 — *Let $\mathfrak{g}$ be a nilpotent Lie algebra and Γ a lattice in the corresponding simply connected Lie group, such that for every complex structure J on $\mathfrak{g}$ the nilmanifold $M = (\mathfrak{g}, J, \Gamma)$ satisfies the conclusion of theorem 3.5. Then every small deformation M' of a nilmanifold $M = (\mathfrak{g}, J, \Gamma)$ is again a nilmanifold with left-invariant complex structure $M' = (\mathfrak{g}, J', \Gamma)$.*

In particular this holds if $\mathfrak{g}$ admits a stable torus bundle series.

In other word, under the above condition the space of left-invariant complex structures is open in the space of all complex structures.

4 Albanese Quotients and deformations in the large

In the whole section Δ will denote a small 1-dimensional disc centred in 0.

In [Cat04] Catanese proved that any deformation in the large of a complex torus is again a complex torus by analysing the Albanese map:

He studies families $\mathcal{X} \to \Delta$ with general fibre a complex torus. In a first step he shows that also the special fibre $\mathcal{X}_0$ has a surjective Albanese map to a complex torus of the same dimension. By proving that this map has to be biholomorphic he concludes that the special fibre is a complex torus.

We will try to generalise his method to the case where the Albanese map is not so well behaved.

4.1 Definitions and results

We need to recall some definitions. Let X be a compact, complex manifold. By Kodaira's Lemma (see [Cat04], Lemma 2.2) we have an inclusion

$$H^0(X, d\mathcal{O}_X) \oplus \overline{H^0(X, d\mathcal{O}_X)} \subset H^1_{dR}(X, \mathbb{C}).$$

Definition 4.1 — *The* ***Albanese variety*** *of X is the abelian, complex Lie group $Alb(X)$ defined as the quotient of $H^0(X, d\mathcal{O}_X)^*$ by the minimal, closed, complex subgroup containing the image of $H_1(X, \mathbb{Z})$ under the map*

$$H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{C}) \to H^0(X, d\mathcal{O}_X)^*.$$

The Albanese variety is called very good if the image of $H_1(X, \mathbb{Z})$ is a lattice in $H^0(X, d\mathcal{O}_X)^*$.

In what follows we are not interested in the torsion part of integral cohomology and will by abuse of notation write $H_1(X, \mathbb{Z})$ for its image in $H_1(X, \mathbb{Q})$.

The Albanese map $\alpha_X : X \to Alb(X)$ is given by integration of closed forms along paths starting from a fixed base point.

Remark 4.2 — Jörg Winkelmann brought to our attention that the Albanese variety is in fact always compact and hence $Alb(X)$ is very good if and only if the image of $H_1(X, \mathbb{Z})$ is discrete in $H^0(X, d\mathcal{O}_X)^*$.

If X satisfies the weak 1-Hodge property which means

$$H^1_{dR}(X, \mathbb{C}) = H^0(X, d\mathcal{O}_X) \oplus \overline{H^0(X, d\mathcal{O}_X)} = H^0(X, d\mathcal{O}_X) \oplus H^1(X, \mathcal{O}_X)$$

then X has a very good Albanese variety.

In particular this holds if X is kählerian because of the Hodge decomposition $H^1(X, \mathbb{C}) = H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X)$. The summands are interchanged by complex conjugation and moreover every holomorphic form is closed.

Catanese studied the behaviour of the Albanese map with respect to deformation and obtained for example the following

Proposition 4.3 ([Cat04], Corollary 2.5) — *Assume that* $\{X_t\}_{t\in\Delta}$ *is a 1-parameter family of compact complex manifolds over the unit disc such that there is a sequence* $t_\nu \to 0$ *with* $\mathcal{X}_{t_\nu}$ *satisfying the weak 1-Hodge property, and moreover such that the image of* X_{t_ν} *under the Albanese map has dimension* a.

Then also the central fibre has a very good Albanese map and Albanese dimension a.

In the following we want to explain how to generalise this result if the Albanese variety is not necessarily very good.

Example 4.4 — Let us for a moment return to our example of a principal holomorphic torus over an elliptic curve $\pi : X = \Gamma\backslash G \to E$ given in 1.13 Let k be the dimension of the fibres. The corresponding Lie algebra with complex structure $(\mathfrak{g}, J)$ has 2 codimensional centre and hence admits a basis $e_1, e_2, z_1, \ldots, z_{2k}$ such that the only non trivial Lie bracket relation is $[e_1, e_2] = z_{2k}$ and $\mathcal{C}^1\mathfrak{g}$ is spanned by z_{2k}.

Note that we proved that the centre $\mathcal{Z}\mathfrak{g} = \langle z_1, \ldots, z_{2k}\rangle$ is invariant under J. Let $e^1, \ldots, z^{2k}$ be the dual basis.

We will use the results mentioned in Section 3 to describe the relevant cohomology groups. The differential $d : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ in the Chevalley complex is given by

$$dz^{2k} = -e^1 \wedge e^2, de^1 = de^2 = dz^1 = \cdots = dz^{2k-1} = 0$$

since it is defined by $d\alpha(x, y) = -\alpha([x, y])$. So

$$H^1(X, \mathbb{C}) = \{\alpha \in \mathfrak{g}^*_{\mathbb{C}} \mid d\alpha = 0\}$$

has dimension $2k + 1$ and X can never satisfy the weak 1-Hodge property. Indeed, $\omega := z^{2k} + iJz^{2k}$ is a complex differential form of type $(1, 0)$ with $\overline{\partial}\omega \neq 0$ and in particular ω is not closed.

Letting $\mathcal{C}^1_J\mathfrak{g} := \mathcal{C}^1\mathfrak{g} + J\mathcal{C}^1\mathfrak{g}$ we have

$$\begin{aligned} H^0(X, d\mathcal{O}_X) &= \{\alpha \in \mathfrak{g}^{*1,0} \mid d\alpha = 0\} \\ &= \{\alpha \in \mathfrak{g}^{*1,0} \mid \overline{\partial}\alpha = 0\} \\ &= Ann((\mathcal{C}^1_J\mathfrak{g}) \otimes \mathbb{C}) \cap \mathfrak{g}^{*1,0} \end{aligned}$$

and

$$\begin{aligned} H^1(X, \mathcal{O}_X) &= H^{0,1}(X) \\ &= \{\bar{\alpha} \in \mathfrak{g}^{*0,1} \mid \overline{\partial}\bar{\alpha} = 0\} \\ &= \mathfrak{g}^{*0,1}. \end{aligned}$$

In order to determine the Albanese Variety of X we have to take the rational structure on $\mathfrak{g}$ induced by the lattice Γ into account; on the quotient $H_1(X,\mathbb{R}) = \mathfrak{g}/\mathcal{C}^1\mathfrak{g}$ it maps to the rational structure induced by $H_1(X,\mathbb{Z})$.

Let W be the smallest J-invariant, Γ-rational subspace of $\mathfrak{g}$ which contains $\mathcal{C}^1\mathfrak{g}$. Since the centre is J-invariant and Γ-rational we have

$$2 \leq d := \dim_{\mathbb{R}} W \leq \dim_{\mathbb{R}} \mathcal{Z}\mathfrak{g} = 2k.$$

The kernel of the surjection $p : H_1(X,\mathbb{R}) \twoheadrightarrow H^0(X, d\mathcal{O}_X)^*$ is $\mathcal{C}^1_J\mathfrak{g}/\mathcal{C}^1\mathfrak{g}$ and hence the Albanese variety of X is exactly

$$Alb(X) = \frac{H^0(X, d\mathcal{O}_X)^*/p(W)}{\operatorname{im}(H_1(X,\mathbb{Z}) \to H^0(X, d\mathcal{O}_X)^*/p(W))}.$$

The dimension of the Albanese variety can vary with the complex structure in the range

$$1 \leq \dim Alb(X) = \frac{\dim_{\mathbb{R}} \mathfrak{g} - d}{2} \leq k = h^0(X, d\mathcal{O}_X).$$

But in some sense all the ugly phenomena happen inside $\mathcal{Z}\mathfrak{g}$. Indeed, if we replace W by $\mathcal{Z}\mathfrak{g}$ in the above construction we get a map to an elliptic curve such that the corresponding map in cohomology is independent of the complex structure. In fact, we have found a way to reconstruct our map $\pi : X \to E$ from cohomological data which does not depend on the particular left-invariant complex structure.

Thus, even if the Albanese map is not well behaved, we might find some subspace in $H_1(X,\mathbb{R})$, not depending on the complex structure, which yields a holomorphic map to a compact torus. We will now formalise this idea.

Let $V \subsetneq H_1(X,\mathbb{Q})$ be a proper subspace and let

$$U := Ann(V_{\mathbb{C}}) \cap H^0(X, d\mathcal{O}_X)$$

where the annullator is taken with respect to the pairing between $H^1(X,\mathbb{C})$ and $H_1(X,\mathbb{C})$. Assuming that

$$Ann(V_{\mathbb{C}}) = U \oplus \bar{U}$$

the map

$$H_1(X,\mathbb{R})/V_{\mathbb{R}} \xrightarrow{\cong} U^*$$

is an isomorphism of real vector spaces.

Since V is a rational subspace $H_1(X,\mathbb{Z})/(H_1(X,\mathbb{Z}) \cap V)$ maps to a full lattice $\Lambda \subset U^*$. This yields a complex torus U^*/Λ which is a quotient of the Albanese variety $Alb(X)$, the projection being induced by the inclusion $U \hookrightarrow H^0(X, d\mathcal{O}_X)$.

We have shown that the following is well defined and yields in fact a compact, complex torus:

Definition 4.5 — *Let $V \subsetneq H_1(X, \mathbb{Q})$ be a proper subspace such that*

$$Ann(V_{\mathbb{C}}) = U \oplus \bar{U}$$

where $U := Ann(V_{\mathbb{C}}) \cap H^0(X, d\mathcal{O}_X)$. Then we call

$$QAlb_V(X) := U^*/\Lambda$$

the ***very good Albanese quotient*** *of X associated to V, where Λ is the image of the lattice $H_1(X,\mathbb{Z})/(H_1(X,\mathbb{Z}) \cap V)$ under the map $H_1(X,\mathbb{R})/V_{\mathbb{R}} \to U^*$.*

Remark 4.6 — The above condition on V is satisfied if and only if the kernel of the composition map

$$\phi : H^0(X, d\mathcal{O}_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C})/Ann(V_{\mathbb{C}})$$

has complex dimension $q = \frac{\dim_{\mathbb{Q}}(V)}{2}$. This is in fact the maximal possible dimension since $V_{\mathbb{C}}$ is defined over $\mathbb{R}$ while on the other hand

$$H^0(X, d\mathcal{O}_X) \cap \overline{H^0(X, d\mathcal{O}_X)} = 0.$$

Another way to look at this is the following: let

$$W := (H^0(X, d\mathcal{O}_X) \oplus \overline{H^0(X, d\mathcal{O}_X)}) \cap H^1(X, \mathbb{R}).$$

The decomposition $W_{\mathbb{C}} = H^0(X, d\mathcal{O}_X) \oplus \overline{H^0(X, d\mathcal{O}_X)}$ defines a complex structure on W and the above conditions on V are equivalent to $Ann(V_{\mathbb{R}})$ being a complex subspace of W with respect to this complex structure.

Therefore the dimension of $Ann(V_{\mathbb{R}})$ is necessarily even.

Remark 4.7 — Integration over the closed forms which are contained in $Ann(V_{\mathbb{C}}) \cap H^0(X, d\mathcal{O}_X)$ gives us a holomorphic map $q\alpha_V$ which factors through the Albanese map:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Alb(X) \\ & \searrow^{q\alpha_V} & \downarrow \\ & & QAlb_V(X). \end{array}$$

Note that $QAlb_V(X)$ has positive dimension by definition since we assumed $V \neq H_1(X, \mathbb{Q})$.

Example 4.8 — Let $M = (\mathfrak{g}, J, \Gamma)$ be a 2-step nilmanifold such that the centre $\mathcal{Z}\mathfrak{g}$ is J-invariant. By Nomizu's theorem we have $H_1(M, \mathbb{R}) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ and $V_{\mathbb{R}} := \mathcal{Z}\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is a Γ-rational subspace. Then $V_{\mathbb{Q}}$ induces a very good Albanese quotient and $q\alpha_V$ coincides with the principal holomorphic bundle map induced by the filtration $\mathfrak{g} \supset \mathcal{Z}\mathfrak{g} \supset 0$. In particular this applies to the example discussed above.

Our goal is to study such locally trivial, smooth, holomorphic fibrations. Hence we define

Definition 4.9 — *Let $\mathcal{C}$ be a class of compact, complex manifolds and let X any compact, complex manifold. We say that a subspace $V \subsetneq H_1(X, \mathbb{Q})$ is* ***$\mathcal{C}$-fibering*** *on X if V induces a very good Albanese quotient and the map*

$$q\alpha : X \to QAlb_V(X)$$

is a locally trivial, smooth, holomorphic fibration such that the fibre is in $\mathcal{C}$.

A class $\mathcal{C}$ of compact complex manifolds is called a ***good fibre class*** *if the following conditions hold*

(*i*) *The class $\mathcal{C}$ is closed under holomorphic 1-parameter limits, i.e., if we have a smooth family over the unit disc and a sequence $(t_\nu)_{\nu\in\mathbb{N}}$ converging to 0 such that the fibres over t_ν are in $\mathcal{C}$ then also the central fibre is in $\mathcal{C}$.*

(*ii*) *One of the following conditions holds:*

(*a*) *There is a coarse moduli space $\mathfrak{M}_{\mathcal{C}}$ for manifolds in class $\mathcal{C}$ which is Hausdorff.*

(*b*) *For every manifold $Y \in \mathcal{C}$ there is a local moduli space, i.e., the Kuranishi family of Y is universal.*

(*c*) *$h^1(F, \Theta_F)$ is constant on the connected components of $\mathcal{C}$.*

So in the above example the subspace V is torus-fibering.

Remark 4.10 — The condition (a) is very strong but it is worth including since it applies to Kodaira surfaces. Indeed these and complex tori are our first examples of good fibre classes.

That every deformation in the large of a complex torus is again a complex torus has been proved by Catanese in [Cat02] and we have $h^1(F, \Theta_F) = \dim(T)^2$ for any Torus T (for example by Section 3).

Borcea proved in [Bor84] that the moduli space of Kodaira surfaces may be identified with the product of the complex plane by the punctured disc which is Hausdorff. That every deformation in the large is again a Kodaira surface has been proved by the author in [Rol05].

We will come back to this issue in Section 6.

The main result of this section is

Theorem 4.11 — *Let $\mathcal{C}$ be a good fibre class and $\pi : \mathcal{X} \to \Delta$ be a smooth family of compact, complex manifolds. Let $(t_\nu)_{\nu\in\mathbb{N}}$ be a sequence in Δ converging to zero.*

If there is a subspace $V \subsetneq H_1(\mathcal{X}_0, \mathbb{Q})$ which is $\mathcal{C}$-fibering on $\mathcal{X}_{t_\nu}$ for all ν then V is $\mathcal{C}$-fibering on $\mathcal{X}_0$.

The proof will be given in the next subsection. We will first show how this can be applied to nilmanifolds

Theorem 4.12 — *Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and let $\Gamma \subset G$ be a lattice such that the following holds:*

(*i*) *$\mathfrak{g}$ admits a stable torus bundle series $(\mathcal{S}^i\mathfrak{g})_{i=0,\dots,t}$ (cf. Definition 1.10).*

(*ii*) *The nilmanifolds of the type $(\mathcal{S}^{t-1}\mathfrak{g}, J, \Gamma \cap \exp(\mathcal{S}^{t-1}\mathfrak{g}))$ constitute a good fibre class.*

Then any deformation in the large M' of a nilmanifold of type $M = (\mathfrak{g}, J, \Gamma)$ is again of the same type $M' = (\mathfrak{g}, J', \Gamma)$.

Proof. We will use the following:

Lemma 4.13 (Lemma 2.8 in [Cat04]) — *Let $\mathcal{B}$ be a connected complex analytic space and $\mathcal{B}'$ a non empty, open subset of $\mathcal{B}$ such that $\mathcal{B}'$ is closed for holomorphic 1-parameter limits (i.e., given any holomorphic map $f : \Delta \to \mathcal{B}$, if there is a sequence $(t_\nu)_{\nu\in\mathbb{N}}$ converging to 0 with $f(t_\nu) \in \mathcal{B}'$ then $f(0) \in \mathcal{B}'$). Then $\mathcal{B} = \mathcal{B}'$.*

Now consider any smooth family $\mathcal{X} \to \mathcal{B}$ over a connected base $\mathcal{B}$ such that for some point $b \in \mathcal{B}$ the fibre $\mathcal{X}_b$ is isomorphic to a nilmanifold $M = (\mathfrak{g}, J, \Gamma)$. Then consider the (non empty) set

$$\mathcal{B}' := \{t \in \mathcal{B} \mid \mathcal{X}_t \cong M' = (\mathfrak{g}, J', \Gamma) \text{ for some complex structure } J'\}.$$

This set is open in view of our result on small deformations in Corollary 3.7. We have to show that it is closed under holomorphic 1-parameter limits.

Note that our definition of stable torus bundle series guarantees that $\Gamma' := \Gamma \cap \exp(\mathcal{S}^{t-1}\mathfrak{g})$ is in fact a lattice. Taking

$$V_\mathbb{R} = \mathcal{S}^{t-1}\mathfrak{g}/\mathcal{C}^1\mathfrak{g} \subset \mathfrak{g}/\mathcal{C}^1\mathfrak{g} \cong H^1(M, \mathbb{R})$$

we can apply Theorem 4.11 which yields that any 1-parameter limit $\hat{M}$ of manifolds in $\mathcal{B}'$ is a smooth, locally trivial, holomorphic fibration over a torus with fibre a nilmanifold $F = (\mathcal{S}^{t-1}\mathfrak{g}, J, \Gamma')$. Since a small deformation of $\hat{M}$ is an iterated principal bundle also $\hat{M}$ itself is an iterated principal bundle hence a nilmanifold.

Note that the fundamental group uniquely determines the Lie algebra and the simply connected Lie group. ([VGS00], p. 45, corollary 2.8.). So $\hat{M} \in \mathcal{B}'$ hence $\mathcal{B}'$ is closed under holomorphic 1-parameter limits and, by the lemma, $\mathcal{B} = \mathcal{B}'$ which concludes the proof. □

Returning to our example we obtain using Remark 4.10:

Corollary 4.14 — *Any deformation in the large of a principal holomorphic torus bundle over an elliptic curve is again a principal holomorphic torus bundle over an elliptic curve.*

4.2 Proof of Theorem 4.11

We will split the proof of the theorem into several steps.

Catanese showed in [Cat91] that the Albanese dimension is in fact a topological property if X is a Kähler manifold and we review his arguments in our context:

Assume that we have X and $V \subset H_1(X, \mathbb{Q})$ as in Definition 4.5.

Lemma 4.15 — *The dimension of the image of X under the map $q\alpha$ is*

$$d = max\{m \mid \operatorname{im}(\Lambda^m(Ann(V_{\mathbb{C}}) \cap H_0(d\mathcal{O}_X)) \to H^m(X, \mathbb{C})) \neq 0\}.$$

Moreover $q\alpha$ is surjective if and only if $\Lambda^{2k} Ann(V_{\mathbb{C}}) \neq 0$ in $H^{2k}(X, \mathbb{C})$ where $k = \dim Ann(V_{\mathbb{C}})$. In particular the surjectivity of $q\alpha$ is a property which depends only on the topology of X and the subspace V.

Proof. By the definition of $QAlb_V(X)$ we have

$$U := q\alpha^* H^0(QAlb_V(X), d\mathcal{O}_{QAlb_V(X)}) = Ann(V) \cap H_0(d\mathcal{O}_X)$$

and $q\alpha$ is given by integration over the holomorhic 1-foms in U.

The dimension of its image is, by Sard's theorem, equal to the maximal rank of the differential of the quotient Albanese map and hence equal to d.

It remains to show that $\Lambda^{2k} Ann(V) \neq 0$ if and only if $\Lambda^k U \neq 0$. But this is clear since our assumptions guarantee that $Ann(V)$ is contained in $H^0(X, d\mathcal{O}_X) \oplus \overline{H^0(X, d\mathcal{O}_X)}$, hence $Ann(V)_{\mathbb{C}} = U \oplus \bar{U}$ and $\Lambda^{2k} Ann(V) = \Lambda^k U \otimes \Lambda^k \bar{U}$. □

We will now analyse how our notion of Albanese quotient behaves under deformation.

Let $\pi : \mathcal{X} \to \Delta$ be a smooth family of compact complex manifolds over the unit disk. We identify $H_1(\mathcal{X}_t, \mathbb{Q})$ with $H_1(\mathcal{X}_0, \mathbb{Q})$.

If $V \subset H_1(\mathcal{X}_0, \mathbb{Q})$ gives rise to a very good Albanese quotient on $\mathcal{X}_0$, the special fibre, then there are examples where this fails for fibres near $\mathcal{X}_0$. (Consider for example a product of two elliptic curves which deforms to a simple torus or see Example 1.14.) But on the other hand the following holds:

Proposition 4.16 — *Let $\pi : \mathcal{X} \to \Delta$ be a smooth family of compact, complex manifolds over the unit disk and let $V \subset H_1(\mathcal{X}_0, \mathbb{Q})$. Then the set*

$$Q := \{t \in \Delta \mid V \textit{ defines a very good Albanese Quotient on } \mathcal{X}_t\}$$

is a (possibly empty) analytic subset of Δ.

Before we come to the proof we will give an application:

Corollary 4.17 — *Let $\pi : \mathcal{X} \to \Delta$ be a smooth family of compact, complex manifolds over the unit disk and let $(t_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging some point in Δ. If $V \subset H_1(\mathcal{X}_0, \mathbb{Q})$ gives rise to a very good Albanese quotient on $\mathcal{X}_t$, for $t = t_\nu$ then this holds for all $t \in \Delta$.*

If the quotient Albanese map is surjective for some t_ν then it is surjective for all t.

Proof. Consider the set Q as in the Proposition. An analytic subset which has an accumulation point must have dimension at least one. But since Δ itself is 1-dimensional we have $Q = \Delta$ and the first claim is proved. The last statement follows immediately from 4.15. □

Now we need to introduce some notation. The sheaves of relative differential forms $\Omega^k_{\mathcal{X}/\Delta}$ are defined by the exact sequence

$$0 \to \pi^*\Omega^k_\Delta \to \Omega^k_{\mathcal{X}} \to \Omega^k_{\mathcal{X}/\Delta} \to 0$$

and we have a $\pi^*\mathcal{O}_\Delta$-linear map

$$d_v : \Omega^{k-1}_{\mathcal{X}/\Delta} \xrightarrow{d} \Omega^k_{\mathcal{X}} \to \Omega^k_{\mathcal{X}/\Delta}$$

given by differentiation along the fibres. This gives us a complex of sheaves

$$\mathcal{E}^\bullet = 0 \to \mathcal{O}_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/\Delta} \to \Omega^2_{\mathcal{X}/\Delta} \to \dots$$

which restricts to the holomorphic de Rham complex on every fibre $\mathcal{X}_t$ of π.

The following useful fact has been extracted from the proof of Lemma 2.4 in [Cat04]. Denote by $\mathbb{C}_t$ the residue field at the point $t \in \Delta$.

Lemma 4.18 — *Let $\pi : \mathcal{X} \to \Delta$ be a smooth family of compact complex manifolds over a small disk and assume that there is a sequence $(t_\nu)_{\nu\in\mathbb{N}}$ converging to 0 such that $H^0(\mathcal{X}_{t_\nu}, d\mathcal{O}_{\mathcal{X}_{t_\nu}})$ has dimension q for all ν.*

Then, after possibly shrinking Δ, there is a rank q locally free subsheaf $\mathcal{H}$ of $H^1(\mathcal{X}_0, \mathbb{C}) \otimes \mathcal{O}_\Delta$, the trivial vector bundle on Δ with fibre $H^1(\mathcal{X}_0, \mathbb{C})$, such that

$$\mathcal{H} \otimes \mathbb{C}_t \subset H^0(\mathcal{X}_t, d\mathcal{O}_{\mathcal{X}_t}) \text{ for all } t \in \Delta$$

and equality holds for $t \neq 0$.

Proof. Let Δ^* be the pointed disk $\Delta \setminus \{0\}$. We define a (not necessarily locally free) sheaf $d\mathcal{O}_{\mathcal{X}/\Delta}$ on $\mathcal{X}$ by the exact sequence

$$0 \to d\mathcal{O}_{\mathcal{X}/\Delta} \to \Omega^1_{\mathcal{X}/\Delta} \xrightarrow{d_v} \Omega^2_{\mathcal{X}/\Delta}$$

and push down this sequence to Δ via π_*.

Since the sheaves $\Omega^k_{\mathcal{X}/\Delta}$ are torsion free the same holds for their direct images $\pi_*\Omega^k_{\mathcal{X}/\Delta}$, and, Δ being smooth of dimension 1, the $\pi_*\Omega^k_{\mathcal{X}/\Delta}$ are in fact locally free. This implies that also $\mathcal{H} := \pi_* d\mathcal{O}_{\mathcal{X}/\Delta}$ is locally free since it is a subsheaf of a locally free sheaf on Δ. Any section in $\mathcal{H}$ is a holomorphic differential form on $\mathcal{X}$ which restricts to a closed form on any fibre.

The base change map

$$\pi_*\Omega^k_{\mathcal{X}/\Delta} \otimes \mathbb{C}_t \hookrightarrow H^0(\mathcal{X}_t, \Omega^k_{\mathcal{X}/\Delta|\mathcal{X}_t}) = H^0(\mathcal{X}_t, \Omega^k_{\mathcal{X}_t})$$

is an injection, since $\{t\}$ has codimension 1 in Δ ([GR84], Prop. 2, p. 208, p. 209). By possibly shrinking our disk we may assume that on Δ^* the dimensions of $H^0(\mathcal{X}_t, \Omega^k_{\mathcal{X}_t})$ are constant ($k = 1, 2$) and the map of vector bundles $d_v : \pi_*\Omega^1_{\mathcal{X}/\Delta} \to \pi_*\Omega^2_{\mathcal{X}/\Delta}$ has constant rank.

Then also $h^0(\mathcal{X}_t, d\mathcal{O}_{\mathcal{X}_t})$ is constant on Δ^* and we have an isomorphism

$$\pi_* d\mathcal{O}_{\mathcal{X}/\Delta} \otimes \mathbb{C}_t \cong H^0(\mathcal{X}_t, d\mathcal{O}_{\mathcal{X}_t})$$

for $t \neq 0$. In particular $\mathcal{H}$ has rank $q = h^0(\mathcal{X}_{t_\nu}, d\mathcal{O}_{\mathcal{X}_{t_\nu}})$.

The map $\mathcal{H} \hookrightarrow H^1(\mathcal{X}_0, \mathbb{C}) \otimes \mathcal{O}_\Delta$ is induced by the inclusion

$$H^0(\mathcal{X}_t, d\mathcal{O}_{\mathcal{X}_t}) \hookrightarrow H^1(\mathcal{X}_0, \mathbb{C})$$

on each fibre. □

Proof of Proposition 4.16. The question is local on Δ and hence we may assume that we are in the situation of the Lemma. Consider the composition map of vector bundles on Δ given by

$$\phi : \mathcal{H} \to H^1(\mathcal{X}_0, \mathbb{C}) \otimes \mathcal{O}_\Delta \to \left(H^1(\mathcal{X}_0, \mathbb{C})/Ann(V_\mathbb{C})\right) \otimes \mathcal{O}_\Delta.$$

Let $\dim_\mathbb{C} Ann(V_\mathbb{C}) = 2q$. Then V induces a very good Quotient Albanese map on $\mathcal{X}_t$ if and only if the kernel of the map ϕ_t has dimension q which is the maximal possible dimension (see Remark 4.6). Writing ϕ as a matrix with holomorphic entries we see that this is equivalent to the vanishing of the determinants of all minors of a certain dimension which is an analytic condition. □

Here comes another application of the lemma:

Lemma 4.19 — *Let $\pi : \mathcal{X} \to \Delta$ be a smooth family over a small disc and let $V \subset H_1(\mathcal{X}_0, \mathbb{Q})$ be a subspace inducing a very good Albanese quotient on every fibre. Then after possibly shrinking Δ there is a family of tori $\pi' : \mathcal{B} \to \Delta$ and a map Φ inducing a diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{B} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & \Delta & \end{array}$$

such that for every $t \in \Delta$ the map $\Phi_t : \mathcal{X}_t \to \mathcal{B}_t$ is the quotient Albanese map.

We say that Φ is a family of Albanese Quotients.

Proof. We may assume that we are in the situation of Lemma 4.18 and that there is a local cross section $s : \Delta \to \mathcal{X}$. We define $\mathcal{H}' := Ann(V_{\mathbb{C}}) \cap \mathcal{H}$ and get our family $\mathcal{B} \to \Delta$ by taking the quotient of $\mathcal{H}'^*$ by the image of $H_1(X, \mathbb{Z})/(H_1(X, \mathbb{Z}) \cap V)$. The map Φ can be defined by mapping a point $x \in \mathcal{X}$ to the map $\omega \mapsto \int_{\gamma_x} \omega$ where γ_x is any path joining x to $s(\Delta)$. Then Φ restricts to the quotient Albanese map on every fibre as claimed. □

Proposition 4.20 — *Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ \Phi\ } & \mathcal{B} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & \Delta & \end{array}$$

be a family of Albanese Quotients such that $\Phi_t : \mathcal{X}_t \to \mathcal{B}_t$ *is a smooth holomorphic fibration for* $t = t_\nu$ *where* $(t_\nu)_{\nu \in \mathbb{N}}$ *is a sequence converging to 0 in* Δ*. Then there is a small neighbourhood* Δ' *of zero such that* Φ_t *is a smooth holomorphic fibration for all* $t \in \Delta'$*.*

Proof. By Proposition 4.17 the map Φ_0 is also surjective. Recall that Φ is given by integration over the closed holomorphic 1-forms in

$$U := Ann(V_{\mathbb{C}}) \cap H_0(d\mathcal{O}_{\mathcal{X}}).$$

Let $\omega_1, \ldots, \omega_m$ be a basis for U. Then $\omega := \omega_1 \wedge \cdots \wedge \omega_m$ generates a sub line bundle of $\Omega^m_{\mathcal{X}}$, namely the pullback $\Phi^* K_{\mathcal{B}/\Delta}$ of the relative canonical bundle of the family $\mathcal{B} \to \Delta$.

The rank of the Jacobian of Φ in some point $p \in \mathcal{X}$ is not maximal if and only if ω vanishes in p, i.e. in the points of the zero divisor $R := Z(\omega)$. But since Φ_t is smooth for $t = t_\nu$ the divisor R is completely contained in a union of fibres and we can choose Δ' such that Φ_t is smooth for $t \in \Delta' \setminus \{0\}$.

Hence we may assume that R is completely contained in $\mathcal{X}_0$ and since $\mathcal{X}_0$ is irreducible of codimension one we have in fact $R = \mathcal{X}_0$ or $R = \emptyset$.

If R is not empty then there is a minimal $k \in \mathbb{N}$ such that ω/t^k is holomorphic and hence, after a base change $\Delta \xrightarrow{t^k} \Delta$, there is at least one point in $\mathcal{X}_0$ where ω does not vanish. But then it can vanish nowhere by dimension reasons and this proves that the Jacobian has maximal rank everywhere and the central fibre is indeed a smooth holomorphic fibration. □

Now that we know that our fibration in the limit is indeed a smooth holomorphic fibration we want to give condition under which it is also locally free provided this holds for Φ_t $(t \neq 0)$.

If $\mathcal{X}$ is a complex manifold we denote by $\Theta_{\mathcal{X}}$ its sheaf of holomorphic tangent vectors. For a family of manifolds $\Phi : \mathcal{X} \to \mathcal{B}$ we define the sheaf of relative tangent vectors $\Theta_{\mathcal{X}/\mathcal{B}}$ by the sequence

$$0 \to \Theta_{\mathcal{X}/\mathcal{B}} \to \Theta_{\mathcal{X}} \to \Phi^*\Theta_{\mathcal{B}} \to 0.$$

Proposition 4.21 — *Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{B} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & \Delta & \end{array}$$

be a family of smooth holomorphic fibrations over tori, parametrised by the unit disc, such that all the fibres of the maps $\Phi_t : \mathcal{X}_t \to \mathcal{B}_t$ are contained in a class $\mathcal{C}$ of compact complex manifolds.

If Φ_t is locally trivial for $t \neq 0$ then also Φ_0 is locally trivial if one of the following conditions holds:

(*i*) *There is a coarse moduli space $\mathfrak{M}_{\mathcal{C}}$ for manifolds in class $\mathcal{C}$ which is Hausdorff.*

(*ii*) *For every manifold in $Y \in \mathcal{C}$ there is a local moduli space, in other words, the Kuranishi family of Y is universal.*

(*iii*) *The sheaf $R^1\Phi_*\Theta_{\mathcal{X}/\mathcal{B}}$ is locally free. (This holds if $h^1(\Phi^{-1}(p), \Theta_{\mathcal{X}_p})$ does not depend on $p \in \mathcal{B}$.)*

Condition (*i*) is very strong but we included it because we have an immediate application (see Remark 4.10).

Note that by a theorem of Wavrik [Wav69] (*ii*) holds if $h^k(\Phi^{-1}(p), \Theta_{\mathcal{X}_p})$ does not depend on $p \in \mathcal{B}$ for $k = 0$ or $k = 1$ and the Kuranishi space of $\Phi^{-1}(p)$ is reduced for all $p \in \mathcal{B}$.

Proof. In the first case let F, F' be two fibres of Φ_0. By [FG65] it suffices to show that $F \cong F'$. By pullback we can obtain two families $\mathcal{F}$, $\mathcal{F}'$ of manifolds in $\mathcal{C}$ parametrised by Δ such that $\mathcal{F}_t$ is a fibre of Φ_t and with central fibres F and F' respectively. Since Φ_t is locally trivial for $t \neq 0$ by assumption the corresponding moduli maps to $\mathfrak{M}_{\mathcal{C}}$ coincide for $t \neq 0$ and since $\mathfrak{M}_{\mathcal{C}}$ was assumed to be Hausdorff they have to coincide also for $t = 0$ thence $F \cong F'$ as claimed.

For the second case we we consider the family $\mathcal{X} \to \mathcal{B}$ in the neighbourhood of some point $p_0 \in \mathcal{B}_0$.

Let $\mathrm{Kur}(\Phi^{-1}(p_0))$ be the Kuranishi family of $\Phi^{-1}(p_0)$ which is universal by assumption, i.e., in some neighbourhood U of p_0 we get a unique moduli map $\mu : U \to \mathrm{Kur}(\Phi^{-1}(p_0))$ such that the point $\mu(p)$ corresponds to the manifold $\Phi^{-1}(p)$.

If we choose U sufficiently small we can find local coordinates $x = (t, y)$ around p_0 such that $\pi'(x) = t$. Since $\Phi_t : \mathcal{X}_t \to \mathcal{B}_t$ is locally trivial for $t \neq 0$

we have

$$\frac{\partial\mu(t,y)}{\partial y} = 0$$

on the dense set where $t \neq 0$ and hence everywhere.

Therefore the moduli map is constant on $U \cap \mathcal{B}_0$ and, since the point $p_0 \in \mathcal{B}_0$ was arbitrary and $\mathcal{B}_0$ is connected, all the fibres of ϕ_0 are isomorphic and Φ_0 is locally trivial by [FG65].

It remains to treat the last case.

The fibration $\Phi_0 : \mathcal{S}_0 \to \mathcal{B}_0$ is locally trivial if and only if the Kodaira-Spencer map

$$\rho_0 : \Theta_{\mathcal{B}_0} \to R^1\Phi_{0*}\Theta_{\mathcal{X}_0/\mathcal{B}_0}$$

vanishes identically. ([GR84], Prop. 1, p. 208).

We want to study the relation between ρ_0 and the Kodaira-Spencer map ρ for the whole family via the base change homomorphism. In particular we are interested in the subsheaf $\Theta_{\mathcal{B}/\Delta} \subset \Theta_{\mathcal{B}}$ that restricts to $\Theta_{\mathcal{B}_t}$ on every fibre of π'.

Let

$$\rho' : \Theta_{\mathcal{B}/\Delta} \to \Theta_{\mathcal{B}} \xrightarrow{\rho} R^1\Phi_*\Theta_{\mathcal{X}/\mathcal{B}}$$

be the composition map which is a map of vector bundles by our assumptions. We claim that ρ' is in fact identically zero:

Let Z be an analytic subspace of $\mathcal{B}$ of codimension 1 and let $\mathcal{I}$ be the corresponding ideal sheaf. Then for any sheaf $\mathcal{G}$ on $\mathcal{X}$ there is the base change map

$$\Phi_{\mathcal{I}} : R^1\Phi_*\mathcal{G}/\mathcal{I} \cdot R^1\Phi_*\mathcal{G} \to R^1\Phi_*(\mathcal{G}|_{\Phi^{-1}Z})$$

which is injective ([GR84], Prop. 2, p. 208, p. 209).

For our subspaces $\mathcal{B}_t$ the naturality of the base change map yields a commutative diagram

$$\begin{array}{ccc} \Theta_{\mathcal{B}/\Delta}|_{\mathcal{B}_t} & \xrightarrow{\rho'} & \left(R^1\Phi_*\Theta_{\mathcal{X}/\mathcal{B}}\right)|_{\mathcal{B}_t} \\ \downarrow\cong & & \downarrow \\ \Theta_{\mathcal{B}_t} & \xrightarrow[\rho_t]{} & R^1\Phi_{t*}\left(\Theta_{\mathcal{X}_t/\mathcal{B}_t}\right). \end{array}$$

If $t \neq 0$ then we have $\rho_t \equiv 0$ since Φ_t is locally trivial. Therefore the map of vector bundles $\rho' \equiv 0$ because it vanishes on the dense open set $\mathcal{B} \setminus \mathcal{B}_0$.

Looking again at the diagram for $t = 0$ we see that also ρ_0 must be zero. Hence Φ_0 is a locally trivial fibration as claimed. □

Proof of Theorem 4.11. Let $\mathcal{C}$ be a good fibre class and $\pi : \mathcal{X} \to \Delta$ be a family of compact, complex manifolds. Let $(t_\nu)_{\nu\in\mathbb{N}}$ be a sequence in Δ converging to zero such that subspace $V \subsetneq H_1(X, \mathbb{Q})$ is $\mathcal{C}$-fibering on $\mathcal{X}_{t_\nu}$ for all ν.

In particular V defines a very good Albanese Quotient on $\mathcal{X}_t$ for $t = t_\nu$ and hence for all $t \in \Delta$ by Corollary 4.17. So we may assume that we are in the situation of Lemma 4.19.

Our assumptions guarantee that we can first apply Proposition 4.20 and then Proposition 4.21. This concludes the proof of the theorem. $\square$

5 Complex structures on certain Lie algebras

The aim of this section is to study the possible complex structures on certain types of Lie algebras. We are particularly interested in the existence of stable torus bundle series in Lie algebras in view of Corollary 3.7 and Theorem 4.12.

Using only the dimensions of the subspaces in the descending and ascending central series we will try to give a complete picture for Lie algebras with commutator subalgebra of dimension at most three. By giving lots of examples we will also show that our classification cannot be improved without considering other properties of the Lie algebras.

In the case $\dim \mathcal{C}^1\mathfrak{g} = 1$ we can show that there is in fact a unique complex structure up to isomorphism.

5.1 Notations and basic results

In the sequel $\mathfrak{g}$ will denote a nilpotent Lie algebra and J a complex structure on $\mathfrak{g}$ which is always assumed to be integrable in the sense of Definition 1.1; in particular $\dim \mathfrak{g}$ is always even.

We continue to use the notation introduced in section 1, see section 1.3 for the notation in the examples.

For later reference we collect some basic facts about complex structures and Lie algebras in the next lemma.

Let $(\mathfrak{g}, J)$ be a real Lie algebra with a complex structure J, i.e., $J^2 = -id_{\mathfrak{g}}$. Recall that for a real vector space V we denote by $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ its complexification and that we have a decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ of $\mathfrak{g}_{\mathbb{C}}$ into the i resp. $-i$ eigenspace of $J : V_{\mathbb{C}} \to V_{\mathbb{C}}$.

Lemma 5.1 — *Let $(\mathfrak{g}, J)$ be a real Lie algebra with a complex structure.*

(*i*) *Let $V \subset \mathfrak{g}$ be a real subspace. Then the following are equivalent:*

- *V is J-invariant.*
- *V is a complex subspace of $(\mathfrak{g}, J)$ considered as a complex vector space.*
- *$V_{\mathbb{C}} = (V_{\mathbb{C}} \cap \mathfrak{g}^{1,0}) \oplus (V_{\mathbb{C}} \cap \mathfrak{g}^{0,1})$.*

(*ii*) *If $V \subset \mathfrak{g}$ is J-invariant then $\dim_{\mathbb{R}} V$ is even. In particular if $V \subset \mathfrak{g}$ is a nontrivial, J-invariant subspace contained in a real 2-dimensional subspace W then we have $V = W$.*

(*iii*) *Let V be a real subspace of $\mathfrak{g}$ then the largest J-invariant subspace of V is*
$$\big((V_{\mathbb{C}} \cap \mathfrak{g}^{1,0}) \oplus (V_{\mathbb{C}} \cap \mathfrak{g}^{0,1})\big) \cap \mathfrak{g}$$
where $\mathfrak{g}$ is identified with $\{x \otimes 1 \mid x \in \mathfrak{g}\} \subset \mathfrak{g}_{\mathbb{C}}$.

(*iv*) *For any $x \in \mathfrak{g}$ holds: $ad_x(-) = [x, -] \neq 0 \Leftrightarrow x \notin \mathcal{Z}\mathfrak{g}$.*

(*v*) *If* $\mathfrak{g}$ *is 2-step nilpotent then* $\mathcal{C}^1\mathfrak{g} \subset \mathcal{Z}\mathfrak{g}$.

(*vi*) *Let* $V \subset \mathfrak{g}$ *be a real subspace with* $\dim_{\mathbb{R}} V = 3$. *If* $W_1, W_2 \subset V$ *are both non trivial, J-invariant subspaces, then* $W_1 = W_2$.

(*vii*) *The complex structure is abelian, i.e.,* $\mathfrak{g}^{1,0}$ *is an abelian subalgebra of* $\mathfrak{g}_{\mathbb{C}}$, *if and only if* $[x, y] = [Jx, Jy]$ *for all* $x, y \in \mathfrak{g}$. *In this case the ascending central series* $(\mathcal{Z}^i\mathfrak{g})$ *is J-invariant.*

(*viii*) *If* $\mathcal{C}^1\mathfrak{g}$ *contains no J-invariant subspace then the complex structure J is abelian.*

Proof. We prove only the last three assertions, the rest being clear.

(*vi*) Both W_1 and W_2 have positive, even real dimension, hence

$$\dim_{\mathbb{R}} W_1 = \dim_{\mathbb{R}} W_2 = 2 \text{ and } W_1 \cap W_2 \neq 0$$

by dimensional reasons. Since the intersection $W_1 \cap W_2$ is again J-invariant it has also dimension 2 by (*ii*) and we have

$$W_1 = W_1 \cap W_2 = W_2$$

as claimed.

(*vii*) This is a straightforward calculation which can be found for example in [MPPS06].

(*viii*) Since we assumed J to be integrable $\mathfrak{g}^{1,0}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and hence

$$\mathcal{C}^1\mathfrak{g}^{1,0} \subset \mathcal{C}^1\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}^{1,0} \subset \mathcal{C}^1\mathfrak{g}.$$

If $\mathcal{C}^1\mathfrak{g}$ contains no J-invariant subspace then $\mathcal{C}^1\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}^{1,0} = 0$ by (*i*).

□

The main tool in our analysis will be the following technical lemma.

Lemma 5.2 — *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra with complex structure J. Let* $\mathcal{Z}_J\mathfrak{g} := \mathcal{Z}\mathfrak{g} + J\mathcal{Z}\mathfrak{g}$ *be the smallest complex subspace of* $\mathfrak{g}$ *which contains the centre,* $\mathcal{W}_i := \mathcal{Z}_J\mathfrak{g} \cap \mathcal{C}^i\mathfrak{g}$ *and*

$$\mathcal{V}^J_{i+1} := [\mathcal{W}_i, \mathfrak{g}] = span_{\mathbb{R}}\{[w, x] \mid w \in \mathcal{W}_i, x \in \mathfrak{g}\}.$$

Then the following holds:

(*i*) $\mathcal{V}^J_i$ *is a (possibly trivial) J-invariant subspace of the i-th commutator subalgebra* $\mathcal{C}^i\mathfrak{g}$ *and we have* $\mathcal{V}^J_{i+1} \subset \mathcal{V}^J_i$.

(*ii*) *If there is some* $x \in \mathcal{W}_i \setminus \mathcal{Z}\mathfrak{g}$, *i.e., such that* $ad_x \neq 0$, *then* $\mathcal{V}^j_{i+1} \neq 0$.

(*iii*) *The centre is J-invariant if and only if* $\mathcal{V}^J := \mathcal{V}_1^J = 0$.

If the complex structure is fixed we will often omit it from the notation.

Proof. We begin with the last assertion: The centre is not fixed by J if and only if for some $z \in \mathcal{Z}\mathfrak{g}$ the element Jz is not in the centre, which means that there exists some $x \in \mathfrak{g}$ such that $[Jz, x] \neq 0$ which is equivalent to $\mathcal{V}^J \neq 0$.

For the first assertion we only have to show that $x \in \mathcal{V}_i^J$ implies $Jx \in \mathcal{V}_i^J$. We will do this on generators of the form $x = [Jz, y]$ for some z in the centre of $\mathfrak{g}$ and $y \in \mathfrak{g}$. The Nijenhuis tensor then implies

$$Jx = J[Jz, y] = [Jz, Jy] - [z, y] - J[z, Jy] = [Jz, Jy] \in \mathcal{V}_i^J.$$

The second assertion follows immediately from the definition of the subspaces $\mathcal{V}_i^J$ and 5.1 (*iv*). □

In some cases the lemma will enable us to prove that there do not exist complex structures on a certain class of Lie algebras with the following argument: we assume the existence of a complex structure and then deduce that some odd-dimensional subspace should be invariant under J which is impossible.

We give two applications of the lemma:

Proposition 5.3 — *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra. Then every integrable complex structure on $\mathfrak{g}$ is nilpotent.*

If in addition $\mathcal{C}^1\mathfrak{g}$ has codimension at most 2 in $\mathfrak{g}$ then there does not exist a complex structure in $\mathfrak{g}$.

Proof. Let J be a complex structure on $\mathfrak{g}$. We will prove the first claim by induction on the dimension of $\mathfrak{g}$. We have to show that for some k we have $\mathcal{T}^k\mathfrak{g} = \mathfrak{g}$ in the minimal torus bundle series or equivalently that $\mathfrak{g}/\mathcal{T}^{k-1}\mathfrak{g}$ is an abelian Lie algebra.

Since $\mathfrak{g}$ is 2-step nilpotent we have

$$\mathfrak{g} \supset \mathcal{Z}\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \supset 0$$

and by Lemma 5.2 either $\mathcal{V}_1$ or $\mathcal{Z}\mathfrak{g}$ is a nontrivial J-invariant subspace of the centre. In particular the centre has dimension at least 2 and if $\dim \mathcal{Z}\mathfrak{g} = 2$ then the centre is J-invariant (5.1 (*ii*)). This proves the claim if $\mathfrak{g}$ has dimension 4 since the quotient is then necessarily abelian.

In higher dimension we have $\mathcal{T}^1\mathfrak{g} \neq 0$. If $\mathfrak{g}/\mathcal{T}^1\mathfrak{g}$ is abelian we are done, else $\mathfrak{g}/\mathcal{T}^1\mathfrak{g}$ is still 2-step nilpotent and we can use the induction hypothesis.

In Proposition 12 in [CFGU00] it is proved that the existence of a nilpotent complex structure implies $\dim \mathcal{C}^1\mathfrak{g} \leq \dim \mathfrak{g} - 3$ which yields the second assertion. □

Proposition 5.4 — *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let m be the smallest even number strictly bigger than the dimension of the centre of $\mathfrak{g}$. Assume that $\mathcal{Z}\mathfrak{g} = \mathcal{C}^1\mathfrak{g}$ and that one of the following holds:*

(*i*) *For every m-dimensional subspace W, which contains the centre, the image of the bilinear map*

$$[\cdot,\cdot] : W \times \mathfrak{g} \to \mathcal{C}^1\mathfrak{g}$$

generates $\mathcal{C}^1\mathfrak{g}$ as a vector space.

(*ii*) *The map $ad_x : \mathfrak{g} \to \mathcal{C}^1\mathfrak{g}$ is surjective for all $x \notin \mathcal{Z}\mathfrak{g}$.*

If $\dim(\mathcal{Z}\mathfrak{g})$ is not even then there does not exist any integrable complex structure on $\mathfrak{g}$ and if $\dim(\mathcal{Z}\mathfrak{g})$ is even then any complex structure on such a $\mathfrak{g}$ is nilpotent and

$$\mathfrak{g} \supset \mathcal{Z}\mathfrak{g} = \mathcal{C}^1\mathfrak{g} \supset 0$$

is a stable torus bundle series for $\mathfrak{g}$.

Proof. Clearly the second condition implies the first one. Now assume we have a complex structure J on $\mathfrak{g}$ such that the centre is not J-invariant. Then $\mathcal{Z}_J\mathfrak{g} := \mathcal{Z}\mathfrak{g} + J\mathcal{Z}\mathfrak{g}$ is an even dimensional subspace such that $\mathcal{Z}\mathfrak{g} \subsetneq \mathcal{Z}_J\mathfrak{g}$ and therefore it has dimension at least m. The subspace $\mathcal{V}^J$ is nonempty and by our assumption it is in fact equal to $\mathcal{C}^1\mathfrak{g} = \mathcal{Z}\mathfrak{g}$. But then Lemma 5.2 implies that $\mathcal{Z}\mathfrak{g} = \mathcal{V}^J$ is J-invariant – a contradiction.

Hence the centre is J-invariant for every complex structure J on $\mathfrak{g}$. Therefore the centre cannot have odd dimension if there exists a complex structure. □

5.2 The case $\dim(\mathcal{C}^1\mathfrak{g}) = 1$

Recall that a Heisenberg algebra H_{2n+1} is a nilpotent Lie algebra which admits a basis $x_1, \ldots, x_n, y_1, \ldots, y_n, c$ such that c is central and the structure equations are

$$\begin{aligned} [x_i, y_i] &= -[y_i, x_i] = c, \qquad i = 1, \ldots n, \\ [x_i, x_j] &= -[y_i, y_j] = 0, \qquad i, j = 1, \ldots n. \end{aligned} \tag{3}$$

Proposition 5.5 — *Let $\mathfrak{g}$ be a real nilpotent Lie algebra with $\dim(\mathcal{C}^1\mathfrak{g}) = 1$. Then $\mathfrak{g}$ is 2-step nilpotent and $\mathfrak{g} \cong H_{2n+1} \oplus \mathbb{R}^m$ is the direct sum of a Heisenberg algebra with an abelian Lie algebra with generators $z_1, \ldots, z_m$.*

If the dimension of the centre of $\mathfrak{g}$ is odd (resp. m is even), then $\mathfrak{g}$ does not admit any integrable complex structure.

If the dimension of the centre is even then for any choice of signs

$$\begin{aligned} Jx_i &:= \pm y_i && (i = 1, \dots n), \\ Jz_{2k-1} &:= z_{2k} && (k = 1, \dots r), \\ Jc &:= z_{2r+1}, && J^2 = -id_{\mathfrak{g}}, \end{aligned} \tag{4}$$

defines a complex structure on $\mathfrak{g}$*. Every complex structure on* $\mathfrak{g}$ *is equivalent to a complex structure of this kind.*

In particular every complex structure is abelian and the ascending central series is a stable torus bundle series for $\mathfrak{g}$*.*

Proof. Let $c \in \mathcal{C}^1\mathfrak{g}$ be a generator of the commutator subalgebra. Considering the Lie bracket as an alternating form on $\mathfrak{g}/\langle c \rangle$ the classification of alternating forms on vector spaces. ([Lan84], XIV,ï¿½9) yields the claimed decomposition.

Now assume that $\mathfrak{g}$ admits an integrable complex structure J. The commutator $\mathcal{C}^1\mathfrak{g}$ cannot contain any complex subspace for dimensional reasons and writing the Nijenhuis tensor as

$$[Jx, Jy] - [x, y] = J([Jx, y] + [x, Jy])$$

we see that the left hand side is in $\mathcal{C}^1\mathfrak{g}$ while the right hand side is not. This yields

$$[Jx, Jy] = [x, y]$$

for all $x, y \in \mathfrak{g}$ and hence any complex structure on $\mathfrak{g}$ is abelian by 5.1 (*vii*).

In particular the centre is J-invariant and there is no complex structure on $\mathfrak{g}$ if $\dim_{\mathbb{R}} \mathcal{Z}\mathfrak{g}$ is odd.

It is a straightforward calculation to show that J as in (4) defines a complex structure and it remains to show that every integrable complex structure can be written in this way with respect to a suitable basis.

Assume that we have $\mathfrak{g} = H_{2n+1} \oplus \mathbb{R}^{2r+1}$ as above and we are given a complex structure J on $\mathfrak{g}$. The centre $\mathcal{Z}\mathfrak{g}$ is J-invariant and we can choose a basis for the centre such that

$$Jz_{2k-1} = z_{2k} \qquad (k = 1, \dots r), \qquad Jc = z_{2r+1}.$$

Let us fix an arbitrary complex subspace V such that $\mathfrak{g} = V \oplus \mathcal{Z}\mathfrak{g}$. The remaining elements of the basis are provided by the following:

Claim: If (V, J) is a real vector space with a complex structure and

$$[-,-] : V \times V \to \mathbb{R} \cdot c$$

a non degenerate alternating bilinear form on V such that $[x, y] = [Jx, Jy]$ for all $x, y \in V$ then there exists a basis x_i, y_i of V which satisfies (3) and (4).

We will prove our claim by induction on the dimension of V. Pick any $a \in V$.

Case 1 If $[a, Ja] \neq 0$ then $[a, Ja] = \lambda c$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Setting $x_1 = a/\sqrt{|\lambda|}$, $y_1 := sign(\lambda)Jx_1$ where $sign(\lambda)$ is the sign of λ we have

$$[x_1, y_1] = c \text{ and } Jx_1 = sign(\lambda)y_1.$$

We then apply the induction hypothesis to the subspace orthogonal to $\langle x_1, y_1 \rangle$ with respect to the Lie bracket.

Case 2 If $[a, Ja] = 0$ there is some $b \in V$ such that $[a, b] = c$. If $[b, Jb] \neq 0$ we can go back to case 1. Hence we may assume that also $[b, Jb] = 0$.

Using $[Ja, Jb] = [a, b]$ we calculate

$$\begin{aligned}[a + Jb, J(a + Jb)] &= [a + Jb, Ja - b] \\ &= [a, Ja] - [a, b] + [Jb, Ja] - [Jb, b] \\ &= -2[a, b] = -2c \neq 0.\end{aligned}$$

This brings us back into case 1.

This concludes the proof. □

5.3 The case $\dim(\mathcal{C}^1\mathfrak{g}) = 2$

Proposition 5.6 — *Let $\mathfrak{g}$ be a nilpotent Lie algebra with $\dim(\mathcal{C}^1\mathfrak{g}) = 2$. Then every complex structure on $\mathfrak{g}$ is nilpotent and we have the following cases.*

(*i*) *If $\mathfrak{g}$ is 3-step nilpotent then the following holds: if one of the subalgebras $\mathcal{Z}^i\mathfrak{g}$ has odd dimension, then $\mathfrak{g}$ does not admit any complex structure. If there exists a complex structure J on $\mathfrak{g}$ then J is abelian and the ascending central series is a stable torus bundle series.*

(*ii*) *If $\mathfrak{g}$ is 2-step nilpotent and $\dim(\mathcal{Z}\mathfrak{g})$ is odd then*

$$0 \subset \mathcal{C}^1\mathfrak{g} \subset \mathfrak{g}$$

is a stable torus bundle series on $\mathfrak{g}$.

(*iii*) *If $\mathfrak{g}$ is 2-step nilpotent and $\dim(\mathcal{Z}\mathfrak{g})$ is even then either*

$$0 \subset \mathcal{C}^1\mathfrak{g} \subset \mathfrak{g} \quad \text{or} \quad 0 \subset \mathcal{Z}\mathfrak{g} \subset \mathfrak{g}$$

is a torus bundle series on $\mathfrak{g}$ but a stable torus bundle series does not exist in general if $\dim(\mathcal{Z}\mathfrak{g}) \geq 4$. If $\dim(\mathcal{Z}\mathfrak{g}) = 2$ the two series coincide and we have a stable torus bundle series.

Proof. Everything follows quite directly from the following lemma:

Lemma 5.7 — *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra with* $\dim(\mathcal{C}^1\mathfrak{g}) = 2$ *and* J *a complex structure on* $\mathfrak{g}$*. Then at least one of the subspaces* $\mathcal{Z}\mathfrak{g}$ *and* $\mathcal{C}^1\mathfrak{g}$ *is* J*-invariant.*

Proof. We have $0 \subset \mathcal{V} \subset \mathcal{C}^1\mathfrak{g}$ and since $\mathcal{V}$ is even dimensional and $\dim(\mathcal{C}^1\mathfrak{g}) = 2$ we have either $\mathcal{V} = 0$ or $\mathcal{V} = \mathcal{C}^1\mathfrak{g}$. An application Lemma 5.2 concludes the proof. □

We now treat the cases separately:

(*i*) Assume that we have a complex structure J on $\mathfrak{g}$ and $\mathfrak{g}$ is 3-step nilpotent. It suffices to show that $\mathcal{C}^1\mathfrak{g}$ is not a J-invariant subspace by 5.1 (*vii*), (*viii*) since $\dim_{\mathbb{R}} \mathcal{C}^1\mathfrak{g} = 2$.

Assume the contrary. Writing $\mathcal{C}^2\mathfrak{g} = \langle c_2 \rangle$ we have then $\mathcal{C}^1 = \langle c_1 := Jc_2, c_2 \rangle$, in particular $\mathcal{W}_1 \neq 0$ in the notation of Lemma 5.2. But c_2 has to be in the image of ad_{c_1} and hence by 5.2 (*iii*) $\mathcal{V}_2 \neq 0$ which is impossible since $\dim \mathcal{C}^2\mathfrak{g} = 1$.

(*ii*) Clearly $\mathcal{Z}\mathfrak{g}$ can never be J-invariant for any complex structure on $\mathfrak{g}$ if $\dim \mathcal{Z}\mathfrak{g}$ is odd and hence the assertion follows from Lemma 5.7.

(*iii*) We observe that 2-nilpotency implies $\mathcal{C}^1\mathfrak{g} \subset \mathcal{Z}\mathfrak{g}$ and hence both subspaces coincide if $\dim \mathcal{C}^1\mathfrak{g} = \dim \mathcal{Z}\mathfrak{g} = 2$. So by Lemma 5.7

$$0 \subset \mathcal{C}^1\mathfrak{g} = \mathcal{Z}\mathfrak{g} \subset \mathfrak{g}$$

is a stable torus bundle series in this case. The remaining assertions follow from Example 5.8 given below.

□

Example 5.8 — Consider the following 10-dimensional Lie algebra given by a basis $\mathfrak{g} = \langle e_1, e_2, e_3, f_1, f_2, f_3, z_1, z_2, c_1, c_2 \rangle$ and the structure equations

$$dc^1 = e^{12} + f^{12}, \qquad dc^2 = e^{13} + f^{13}$$

with respect to the dual basis. $\mathfrak{g}$ is a 2-step nilpotent Lie algebra with centre $\mathcal{Z}\mathfrak{g} = \langle z_1, z_2, c_1, c_2 \rangle$ and commutator $\mathcal{C}^1\mathfrak{g} = \langle c_1, c_2 \rangle$. We give three complex structures J_1, J_2 and J_3 on $\mathfrak{g}$ by

$$\begin{aligned}
J_1 e_1 &= f_1, & J_2 z_1 &= e_1, & J_3 e_1 &= f_1,\\
J_1 e_2 &= f_2, & J_2 z_2 &= f_1, & J_3 e_2 &= f_2,\\
J_1 e_3 &= f_3, & J_2 c_1 &= c_2, & J_3 e_3 &= f_3,\\
J_1 z_1 &= c_1, & J_2 e_2 &= e_3, & J_3 z_1 &= z_2,\\
J_1 z_2 &= c_2, & J_2 f_2 &= f_3, & J_3 c_1 &= c_2.
\end{aligned}$$

One can check that all these are fact integrable. The complex structure J_1 leaves the commutator invariant but not the centre ($\mathcal{V} = \mathcal{C}^1\mathfrak{g}$), J_2 the centre but not the commutator while J_3 leaves both subspaces invariant. This realises all possible combinations in (*iii*) of the above proposition.

5.4 The case $\dim(\mathcal{C}^1\mathfrak{g}) = 3$

Proposition 5.9 — *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra with* $\dim(\mathcal{C}^1\mathfrak{g}) = 3$. *Then the following cases can occur:*

(*i*) *If* $\mathfrak{g}$ *is 4-step nilpotent then the following holds: if one of the subalgebras* $\mathcal{Z}^i\mathfrak{g}$ *has odd dimension, then* $\mathfrak{g}$ *does not admit any complex structure. If there exists a complex structure* J *on* $\mathfrak{g}$ *then* J *is abelian and the ascending central series is a stable torus bundle series.*

(*ii*) *If* $\mathfrak{g}$ *is 3-step nilpotent and* $\dim(\mathcal{C}^2\mathfrak{g}) = \dim(\mathcal{C}^1\mathfrak{g}\cap\mathcal{Z}\mathfrak{g}) = 1$ *then for any complex structure* J *on* $\mathfrak{g}$ *we have* $\mathcal{C}^2\mathfrak{g}\cap\mathcal{V}^J = 0$ *but* $\mathcal{V}^J \neq 0$ *is possible and there may be nilpotent and non nilpotent complex structures on the same Lie algebra.*

(*iii*) *If* $\mathfrak{g}$ *is 3-step nilpotent such that* $\dim(\mathcal{C}^1\mathfrak{g}\cap\mathcal{Z}\mathfrak{g}) = 2$ *then:*

  (*a*) *If* $\dim\mathcal{Z}^i\mathfrak{g}$ *is even for all* i *then every complex structure* J *on* $\mathfrak{g}$ *is nilpotent and one of the subspaces* $\mathcal{Z}\mathfrak{g}$ *and* $\mathcal{C}^1\mathfrak{g}\cap\mathcal{Z}\mathfrak{g}$ *is* J*-invariant but there is no stable torus bundle series in general.*

  (*b*) *If* $\dim\mathcal{Z}^1\mathfrak{g}$ *is odd or equal to 2 and* $\dim\mathcal{Z}^2\mathfrak{g}$ *is even then*
$$\mathfrak{g}\supset\mathcal{Z}^2\mathfrak{g}\supset\mathcal{C}^1\mathfrak{g}\cap\mathcal{Z}\mathfrak{g}\supset 0$$
  is a stable torus bundle series for $\mathfrak{g}$.

  (*c*) *If* $\dim\mathcal{Z}^2\mathfrak{g}$ *is odd there does not exist any complex structure on* $\mathfrak{g}$.

(*iv*) *If* $\mathfrak{g}$ *is 2-step nilpotent then every complex structure on* $\mathfrak{g}$ *is nilpotent but in general there does not exist a stable torus bundle series.*

Proof. The last assertion follows immediately from Proposition 5.3 and Example 1.14. The remaining cases (*i*), (*i*) and (*iii*) will be treated separately in the following lemmas. □

Lemma 5.10 — *Let* $\mathfrak{g}$ *be a 4-step nilpotent Lie algebra with* $\dim(\mathcal{C}^1\mathfrak{g}) = 3$. *If one of the subalgebras* $\mathcal{Z}^i\mathfrak{g}$ *has odd dimension, then* $\mathfrak{g}$ *does not admit any complex structure. If there exists a complex structure* J *on* $\mathfrak{g}$ *then* J *is abelian and the ascending central series is a stable torus bundle series.*

Proof. Assume that $\mathfrak{g}$ admits a complex structure J. It suffices to show that $\mathcal{C}^1\mathfrak{g}$ does not contain a J-invariant subspace by 5.1 (*vii*), (*viii*).

Assume the contrary, i.e., that there is a non-trivial, J-invariant subspace $W \subset \mathcal{C}^1\mathfrak{g}$ with (necessarily) $\dim_{\mathbb{R}}(W) = 2$. By 5.1 (*vi*) every complex subspace of $\mathcal{C}^1\mathfrak{g}$ is contained in W and equal to W if it is non trivial. Note also that $\dim \mathcal{C}^1\mathfrak{g} = 3$ and $\mathfrak{g}$ 4-step nilpotent imply

$$\dim \mathcal{C}^3\mathfrak{g} = 1, \qquad \dim \mathcal{C}^2\mathfrak{g} = 2,$$
$$\mathcal{Z}\mathfrak{g} \cap \mathcal{C}^1\mathfrak{g} = \mathcal{C}^3\mathfrak{g}.$$

We will now derive a contradiction.

Case 1: $W \cap \mathcal{C}^3\mathfrak{g} = \langle c_3 \rangle \neq 0$ Then $Jc_3 \in \mathcal{C}^1\mathfrak{g} \setminus \mathcal{Z}\mathfrak{g}$ and $\mathcal{V}_2 \neq 0$ by Lemma 5.2 (*ii*). But this in turn implies $\mathcal{C}^2\mathfrak{g} = \mathcal{V}_2 = W$ for dimensional reasons. Hence $Jc_3 \in \mathcal{C}^2\mathfrak{g} \setminus \mathcal{Z}\mathfrak{g}$ and again by 5.2 we have $0 \neq \mathcal{V}_3 \subset \mathcal{C}^3\mathfrak{g}$ which is impossible since $\mathcal{V}_3$ has dimension at least 2.

Case 2: $W \cap \mathcal{C}^3\mathfrak{g} = 0$ We construct a basis of $\mathcal{C}^1\mathfrak{g}$ in the following way: for dimensional reasons W and $\mathcal{C}^2\mathfrak{g}$ intersect non trivially and we can choose a nonzero $c_2 \in W \cap \mathcal{C}^2\mathfrak{g}$. Let $c_1 := Jc_2$ and $c_3 \in \mathcal{C}^3\mathfrak{g}$ be a generator. Then by our assumptions c_1, c_2, c_3 is a basis of $\mathcal{C}^1\mathfrak{g}$.

There is $f_2 \in \mathfrak{g}$ such that $[c_2, f_2] = c_3$. The Nijenhuis tensor together with $Jc_2 = -c_1$ yields

$$J(\underbrace{[Jc_2, f_2] + [c_2, Jf_2]}_{\in \mathcal{C}^2\mathfrak{g}}) = [Jc_2, Jf_2] - [c_2, f_2] \in \mathcal{C}^2\mathfrak{g}$$

and, since $\mathcal{C}^2\mathfrak{g} \cap J\mathcal{C}^2\mathfrak{g} = 0$ under our conditions, both sides are equal to zero. In particular

$$[c_1, f_2] = [Jc_2, f_2] = -[c_2, Jf_2] \in \mathcal{C}^3\mathfrak{g} \tag{5}$$

is central. Writing $c_1 = [a, b]$ we also have

$$[c_1, c_2] = [[a, b], c_2] = [a, [b, c_2]] - [b, [a, c_2]] \in \mathcal{C}^4\mathfrak{g} = 0. \tag{6}$$

Since $c_2 \in \mathcal{C}^2\mathfrak{g} \setminus \mathcal{Z}\mathfrak{g}$ we can find $f_1 \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$ such that

$$[c_1, f_1] = c_2 + \lambda c_3. \tag{7}$$

Writing $[f_1, f_2] = \sum_{i=1}^{3} \lambda_i c_i$ and using the Jacobi identity we get

$$\begin{aligned} c_3 &= [c_2, f_2] = [c_2 + \lambda c_3, f_2] \\ &\overset{(7)}{=} [[c_1, f_1], f_2] \\ &= [c_1, [f_1, f_2]] - [f_1, [c_1, f_2]] \\ &\overset{(5)}{=} [c_1, [f_1, f_2]] \\ &= [c_1, \sum_{i=1}^{3} \lambda_i c_i] \\ &= \lambda_2 [c_1, c_2] \overset{(6)}{=} 0. \end{aligned}$$

This is a contradiction.

Thus we have shown that $\mathcal{C}^1\mathfrak{g}$ cannot contain any J-invariant subspace which implies in turn that any complex structure will be abelian (Lemma 5.1(*viii*)). □

Lemma 5.11 — *Let* $\mathfrak{g}$ *be a 3-step nilpotent Lie algebra with* $\dim(\mathcal{C}^1\mathfrak{g}) = 3$ *and* $\dim(\mathcal{C}^2\mathfrak{g}) = \dim(\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}) = 1$. *Assume that* J *is a complex structure on* $\mathfrak{g}$. *Then* $\mathcal{C}^2\mathfrak{g} \cap \mathcal{V}^J = 0$ *but* $\mathcal{V}^J \neq 0$ *is possible and there may be nilpotent and non nilpotent complex structures on the same Lie algebra.*

Proof. Assume $\mathcal{C}^2\mathfrak{g} \cap \mathcal{V} \neq 0$. Then $J(\mathcal{Z}\mathfrak{g} \cap \mathcal{C}^1\mathfrak{g}) \neq 0$ and hence $\mathcal{V}_2 \neq 0$ which is impossible since $\dim \mathcal{C}^2\mathfrak{g} = 1$. We prove the second assertion in the following example. □

Example 5.12 — Consider the following 10-dimensional Lie algebra given by a basis $\mathfrak{g} = \langle e_1, e_2, e_3, e_4, e_5, e_6, z, c_1, c_2, c_3 \rangle$ and the structure equations

$$\begin{aligned} dc^1 &= e^{15} + e^{16} + e^{35} + e^{36} \\ dc^2 &= e^{25} + e^{26} + e^{45} + e^{46} \\ dc^3 &= e^1 \wedge c^1 + e^3 \wedge c^1 + e^2 \wedge c^2 + e^4 \wedge c^2 \end{aligned}$$

with respect to the dual basis. One can check that $d^2 = 0$, i.e., we have in fact a Lie algebra. The central filtrations are given by

$$\begin{aligned} \mathfrak{g} \supset \mathcal{Z}^2\mathfrak{g} = \langle c_1, c_2, c_3, z \rangle \supset \mathcal{Z}^1\mathfrak{g} = \langle c_3, z \rangle \supset 0 \\ \mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} = \langle c_1, c_2, c_3 \rangle \supset \mathcal{C}^2\mathfrak{g} = \langle c_3 \rangle \supset 0 \end{aligned}$$

and hence we have $\dim \mathcal{C}^1\mathfrak{g} = 3$, $\dim(\mathcal{Z}\mathfrak{g} \cap \mathcal{C}^1\mathfrak{g}) = \dim \mathcal{C}^2\mathfrak{g} = 1$. We give two complex structures J_1 and J_2 on $\mathfrak{g}$ by

$$\begin{aligned} J_1e_1 &= e_2, & J_2e_1 &= e_2,\\ J_1e_3 &= e_4, & J_2e_3 &= e_4,\\ J_1e_5 &= e_6, & J_2e_5 &= z,\\ J_1c_1 &= c_2, & J_2c_1 &= c_2,\\ J_1z &= c_3, & J_2e_6 &= c_3. \end{aligned}$$

A straightforward calculation shows that both structures are integrable and we have

$$\mathcal{V}^{J_1} = 0, \qquad \mathcal{V}^{J_2} = \langle c_1, c_2 \rangle$$

which realises the two possibilities given in the proposition. Note also that J_1 is nilpotent while J_2 is not nilpotent.

Lemma 5.13 — *Let $\mathfrak{g}$ be a 3-step nilpotent Lie algebra with $\dim(\mathcal{C}^1\mathfrak{g}) = 3$ and $\dim(\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}) = 2$.*

- *(i) If $\dim \mathcal{Z}^i\mathfrak{g}$ is even for all i, then every complex structure J on $\mathfrak{g}$ is nilpotent and one of the subspaces $\mathcal{Z}\mathfrak{g}$ and $\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}$ is J-invariant but there is no stable torus bundle series in general.*
- *(ii) If $\dim \mathcal{Z}^1\mathfrak{g}$ is odd or equal to 2 and $\dim \mathcal{Z}^2\mathfrak{g}$ is even then*

$$\mathfrak{g} \supset \mathcal{Z}^2\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g} \supset 0$$

is a stable torus bundle series for $\mathfrak{g}$.

- *(iii) If $\dim \mathcal{Z}^2\mathfrak{g}$ is odd there does not exist any complex structure on $\mathfrak{g}$.*

Proof. Let $\mathfrak{g}$ be as above and assume that J is a complex structure on $\mathfrak{g}$. Note that for dimension reasons any two nontrivial J-invariant subspaces of $\mathcal{C}^1\mathfrak{g}$ coincide (see 5.1 (*vi*)).

As a first step we show that $\mathcal{V} \subset \mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}$.

Assume the contrary. For dimensional reasons we certainly have

$$\mathcal{V} \cap (\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}) \neq 0$$

so that we find some $z \in \mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}$ such that $Jz \in \mathcal{C}^1\mathfrak{g} \setminus (\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g})$, i.e., $Jz \in \mathcal{W}_1 \setminus \mathcal{Z}\mathfrak{g}$ in the notation of Lemma 5.2 (*ii*). Thus $\mathcal{V}_2 \neq 0$ which is a contradiction if $\dim \mathcal{C}^2\mathfrak{g} = 1$.

Else, if $\dim \mathcal{C}^2\mathfrak{g} = 2$, both J-invariant subspaces coincide and

$$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{C}^2\mathfrak{g} = \mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}$$

which contradicts our assumption $\mathcal{V} \not\subset \mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}$.

Therefore we have $\mathcal{V} \subset \mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}$ and at least one of the subspaces $\mathcal{Z}\mathfrak{g}$ and $\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}$ will be J-invariant. Note that the quotient of $\mathfrak{g}$ by any of these subspaces (which are ideals since they are contained in the centre) is a Lie algebra with one dimensional commutator subalgebra. These have been studied in section 5.2 and admit only abelian complex structures if the centre is evendimensional and no complex structures if it has odd dimension. This already proves (iii) and together with Example 5.14 also (i).

If $\dim \mathcal{Z}\mathfrak{g} = 2$ the two subspaces coincide and if $\dim \mathcal{Z}\mathfrak{g}$ is odd then $\mathcal{Z}\mathfrak{g}$ can never be invariant whence(ii). □

In the following two examples we will show that all possibilities in (i) can indeed occur in one Lie algebra and that we cannot achieve a better result by distinguishing the cases $\dim \mathcal{C}^2\mathfrak{g} = 2$ and $\dim \mathcal{C}^2\mathfrak{g} = 1$

Example 5.14 — Consider the following 8-dimensional 3-step nilpotent Lie algebra given by a basis $\mathfrak{g} = \langle e_1, e_2, e_3, c_1, c_2, c_3, z_1, z_2 \rangle$ and the structure equations

$$\begin{aligned} dc^1 &= e^{12} \\ dc^2 &= e^1 \wedge c^1 + e^{23} \\ dc^3 &= e^2 \wedge c^1 - e^{13} \end{aligned}$$

with respect to the dual basis. This defines a Lie algebra since $d^2 = 0$. The central filtrations are given by

$$\mathfrak{g} \supset \mathcal{Z}^2\mathfrak{g} = \langle e_3, c_1, c_2, c_3, z_1, z_2 \rangle \supset \mathcal{Z}^1\mathfrak{g} = \langle c_2, c_3, z_1, z_2 \rangle \supset 0$$
$$\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} = \langle c_1, c_2, c_3 \rangle \supset \mathcal{C}^2\mathfrak{g} = \langle c_2, c_3 \rangle \supset 0$$

We give three complex structures J_1, J_2 and J_3 on $\mathfrak{g}$ by

$$\begin{array}{lll} J_1 e_1 = e_2, & J_2 e_1 = e_2, & J_3 e_1 = e_2, \\ J_1 c_1 = e_3, & J_2 c_1 = e_3, & J_3 c_2 = c_3, \\ J_1 c_3 = c_3, & J_2 c_2 = z_1, & J_3 z_1 = c_1, \\ J_1 z_1 = z_2, & J_2 c_3 = z_2, & J_3 z_2 = e_3. \end{array}$$

It is a straightforward calculation to check that these structures are integrable. The complex structure J_1 leaves both the commutator and the centre invariant, J_2 the centre but not the commutator, while J_3 leaves the commutator invariant but not the centre ($\mathcal{V} = \mathcal{C}^2\mathfrak{g}$). This realises all possible combinations in (i) of the above Lemma 5.13 in a single Lie algebra.

Example 5.15 — Consider the following 18-dimensional Lie algebra given by a basis $\mathfrak{g} = \langle e_1, e_2, e_3, f_1, f_2, f_3, g_1, g_2, g_3, h_1, h_2, h_3, z_1, z_2, c_0, \tilde{c}_0, c_1, c_2 \rangle$ and the structure equations

$$dc^0 = g^{12} + g^{23} + h^{12} + h^{23}$$
$$dc^1 = e^{12} + f^{12}$$
$$\begin{aligned} dc^2 =& e^{13} + f^{13} \\ &+ g^1 \wedge c^0 + c^0 \wedge g^3 + h^1 \wedge c^0 + c^0 \wedge h^3 \\ &- g^1 \wedge \tilde{c}^0 - \tilde{c}^0 \wedge g^3 + h^1 \wedge \tilde{c}^0 + \tilde{c}^0 \wedge h^3 \end{aligned}$$

with respect to the dual basis. The central filtrations are given by

$$\mathfrak{g} \supset \mathcal{Z}^2\mathfrak{g} = \langle e_1, e_2, e_3, f_1, f_2, f_3, c_0, \tilde{c}_0, c_1, c_2, z_1, z_2 \rangle \supset \mathcal{Z}^1\mathfrak{g} = \langle c_1, c_2, z_1, z_2 \rangle \supset 0$$
$$\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} = \langle c_0, c_1, c_2 \rangle \supset \mathcal{C}^2\mathfrak{g} = \langle c_1, c_2 \rangle \supset 0$$

and we have $\dim(\mathcal{C}^1\mathfrak{g}) = 3$, $\dim(\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}) = 2$ and $\dim(\mathcal{C}^2\mathfrak{g}) = 1$.

We already studied the subalgebra

$$\mathfrak{a} = \langle e_1, e_2, e_3, f_1, f_2, f_3, z_1, z_2, c_1, c_2 \rangle$$

in Example 5.8 and in fact we have an extension

$$0 \to \mathfrak{a} \to \mathfrak{g} \to H_7 \oplus \mathbb{R} \to 0.$$

On the subspace $V = \langle g_1, g_2, g_3, h_1, h_2, h_3, c_0, \tilde{c}_0 \rangle$ complementary to $\mathfrak{a}$ we give a complex structure J by

$$Jc_0 = \tilde{c}_0, \qquad Jg_i = h_i, \quad i = 1, \ldots, 3.$$

Note that for $x, y \in V$ we have

$$[x, y] = [Jx, Jy] \text{ and } [x, Jy] = -[Jx, y]$$

and hence the Nijenhuis tensor $[x, y] - [Jx, Jy] + J([Jx, y] + [x, Jy])$ vanishes automatically for these elements in whatever way we choose to extend J to a complex structure on $\mathfrak{g}$. Furthermore we have $[V, \mathfrak{a}] = 0$ and therefore any integrable complex structure on $\mathfrak{a}$ can be combined with J to define an integrable complex structure on $\mathfrak{g}$.

Combining J with J_1,J_2 and J_3 from Example 5.8 we get three integrable complex structures on $\mathfrak{g}$ which realise the different possibilities in (i) of Lemma 5.13.

6 Applications

In this section we put together the results obtained so far. In particular we produce a number of classes of nilmanifolds which are closed under deformation in the large and completely describe the situation in real dimension six.

6.1 The Main Theorem

We want to apply Corollary 3.7 and Theorem 4.12 to our results obtained in Section 5. The outcome is the following:

Theorem 6.1 — *Let $M = (\mathfrak{g}, J, \Gamma)$ a nilmanifold with left-invariant complex structure.*

(*i*) *Any small deformation of M is again a nilmanifold with left-invariant complex structure of the form $M' = (\mathfrak{g}, J', \Gamma)$ if one of the following conditions holds*

- $\dim \mathcal{C}^1\mathfrak{g} \leq 2$.
- $\dim \mathcal{C}^1\mathfrak{g} = 3$ *and* $\mathfrak{g}$ *is 4-step nilpotent.*
- $\dim \mathcal{C}^1\mathfrak{g} = 3$, $\mathfrak{g}$ *is 3-step nilpotent and* $\dim(\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}) = 2$.

(*ii*) *Any deformation in the large of M is again a nilmanifold with left-invariant complex structure of the form $M' = (\mathfrak{g}, J', \Gamma)$ if one of the following conditions holds:*

- $\mathfrak{g}$ *is abelian.*
- $\dim \mathcal{C}^1\mathfrak{g} = 1$.
- $\mathfrak{g}$ *satisfies the conditions of Proposition 5.4.*
- $\dim \mathcal{C}^1\mathfrak{g} = 2$, $\mathfrak{g}$ *is 2-step nilpotent and* $\dim(\mathcal{Z}\mathfrak{g})$ *is equal to 2 or odd.*
- $\mathfrak{g}$ *is 2-step nilpotent and admits a stable torus bundle series of the form* $\mathfrak{g} \supset \mathcal{S}^1\mathfrak{g} \supset 0$.
- $\dim \mathcal{C}^1\mathfrak{g} = 2$ *and* $\mathfrak{g}$ *is 3-step nilpotent.*
- $\dim \mathcal{C}^1\mathfrak{g} = 3$, $\mathfrak{g}$ *is 3-step nilpotent and furthermore*
  - $\dim(\mathcal{C}^1\mathfrak{g} \cap \mathcal{Z}\mathfrak{g}) = 2$
  - $\dim \mathcal{Z}^1\mathfrak{g}$ *is odd or equal to 2.*
  - $\dim \mathcal{Z}^2\mathfrak{g}$ *is even.*
  - $\dim \mathcal{C}^1(\mathcal{Z}^2\mathfrak{g}) = 1$.

  The last condition is automatically fulfilled if $\dim \mathcal{C}^2\mathfrak{g} = 1$.
- $\mathfrak{g}$ *is 3-step nilpotent and admits a stable torus bundle series of the form* $\mathfrak{g} \supset \mathcal{S}^2\mathfrak{g} \supset \mathcal{S}^1\mathfrak{g} \supset 0$ *such that* $\dim(\mathcal{C}^1(\mathcal{S}^2\mathfrak{g})) = 1$.

Moreover in all the above cases both M and M' have the structure of an iterated principal holomorphic torus bundle with fibre dimensions depending only on $\mathfrak{g}$ in the cases described in (ii).

Proof. For the first part it suffices, by Corollary 3.7, to show that under the given conditions $M = (\mathfrak{g}, J, \Gamma)$ is an iterated principal holomorphic torus bundle independently of J. But we showed in Proposition 5.5, 5.6 and 5.9 that there are torus bundle series which are rational with respect to any lattice under these conditions, and this (i).

For (ii), by Theorem 4.12, we have to check that in each of the cases there is a stable torus bundle series $(\mathcal{S}^i\mathfrak{g})_{i=0,\dots,t}$ and that the nilmanifolds of the type $(\mathcal{S}^{t-1}\mathfrak{g}, J, \Gamma \cap \exp(\mathcal{S}^{t-1}\mathfrak{g}))$ constitute a good fibre class. The first assertion has been proved in Proposition 5.4, 5.5, 5.6 and 5.9 or holds by assumption.

If $t = 1$ then M is a torus and the claim follows from [Cat04]. If $t = 2$, i.e., $\mathfrak{g}$ admits a stable torus bundle series

$$\mathfrak{g} \supset \mathcal{S}^1\mathfrak{g} \supset 0$$

then M is a principal holomorphic torus bundle over a torus and we are done, since tori constitute a good fibre class. This is always the case if $\dim \mathcal{C}^1\mathfrak{g} \leq 2$ and $\mathfrak{g}$ is not 3-step nilpotent and if $\mathfrak{g}$ satisfies the conditions of Proposition 5.4.

For the next step we need a lemma:

Lemma 6.2 — *If $\mathfrak{h}$ is a nilpotent Lie algebra with $\dim \mathcal{C}^1\mathfrak{h} = 1$ then the nilmanifolds of type $(\mathfrak{h}, J', \Gamma')$ constitute a closed fibre class.*

Proof. We have already proved above that every deformation in the large of such a manifold is again of this type. Recall from Proposition 5.5 that all complex structures on $\mathfrak{g}$ are equivalent. Hence $h^1(X, \Theta_X)$ is independent of the complex structure J' because $H^1(X, \Theta_X) = H^1_{\bar{\partial}}(\mathfrak{g}, \mathfrak{g})$ can be calculated via Lie algebra Dolbeault cohomology. This is a sufficent condition for a good fibre class. □

If $\mathfrak{g}$ is 3-step nilpotent and has a stable torus bundle series $(\mathcal{S}^i\mathfrak{g})_{i=0,\dots,3}$, then we have to show that the nilmanifolds with Lie algebra $\mathcal{S}^2\mathfrak{g}$ constitute a closed fibre class.

By the lemma it suffices to check that $\dim \mathcal{C}^1(\mathcal{S}^2\mathfrak{g}) = 1$ in the remaining cases. For a 3-step nilpotent Lie algebra we have

$$\mathcal{C}^1(\mathcal{S}^2\mathfrak{g}) \subset \mathcal{C}^1(\mathcal{Z}^2\mathfrak{g}) \subset \mathcal{C}^2\mathfrak{g}$$

and are done if $\dim \mathcal{C}^2\mathfrak{g} = 1$. In the remaining cases the property holds by assumption. This concludes the proof. □

6.2 Deformations and geometric structure in dimension three

In this section we give a fairly complete classification of the geometric types of nilmanifolds with left-invariant complex structure in complex dimension three and determine their deformations.

In [Mag86] Magnin gave a classification of real nilpotent Lie algebra in dimension at most 7 and in particular showed that in real dimension 6 there exist only 34 different isomorphism types while in higher dimension there are always continous families.

Salamon showed in [Sal01] that only 18 of these 6-dimensional real nilpotent Lie algebras admit a left-invariant complex structure and Ugarte ([Uga04] Theorem. 2.9) studied in detail the possible nilpotent and abelian structures. A part of these results has been reproved in Section 5. Following Ugarte's notation we give the list of six dimensional real Lie algebras admitting complex structures:

$$
\begin{aligned}
\mathfrak{h}_1 &= (0,0,0,0,0,0), & \mathfrak{h}_{10} &= (0,0,0,12,13,14),\\
\mathfrak{h}_2 &= (0,0,0,0,12,34), & \mathfrak{h}_{11} &= (0,0,0,12,13,14+23),\\
\mathfrak{h}_3 &= (0,0,0,0,0,12+34), & \mathfrak{h}_{12} &= (0,0,0,12,13,24),\\
\mathfrak{h}_4 &= (0,0,0,0,12,14+23), & \mathfrak{h}_{13} &= (0,0,0,12,13+14,24),\\
\mathfrak{h}_5 &= (0,0,0,0,13+42,14+23), & \mathfrak{h}_{14} &= (0,0,0,12,14,13+42),\\
\mathfrak{h}_6 &= (0,0,0,0,12,13), & \mathfrak{h}_{15} &= (0,0,0,12,13+42,14+23),\\
\mathfrak{h}_7 &= (0,0,0,12,13,23), & \mathfrak{h}_{16} &= (0,0,0,12,14,24),\\
\mathfrak{h}_8 &= (0,0,0,0,0,12), & \mathfrak{h}_{19}^- &= (0,0,0,12,23,14-35),\\
\mathfrak{h}_9 &= (0,0,0,0,12,14+25), & \mathfrak{h}_{26}^+ &= (0,0,12,13,23,14+25).
\end{aligned}
$$

We have seen in example 1.14 that we cannot expect a small deformation of a principal holomorphic torus bundle to carry the same structure even if the complex structure remains nilpotent. Luckily in real dimension six $\mathfrak{h}_7$ is the only case in which such behaviour occurs.

A Kodaira surface is a principal holomorphic fibration of elliptic curves over an elliptic curve. The underlying Lie algebra can be described as $(0,0,0,12)$.

Theorem 6.3 — *Let $M = (\mathfrak{g}, J, \Gamma)$ be a complex 3-dimensional nilmanifold with left-invariant complex structure. If $\mathfrak{g}$ is not in $\{\mathfrak{h}_7, \mathfrak{h}_{19}^-, \mathfrak{h}_{26}^+\}$, then M has the structure of an iterated principal holomorphic torus bundle. We list the possibilities in the following table:*

base	*fibre*	*corresponding Lie algebras*
3-torus	-	$\mathfrak{h}_1$
2-torus	*elliptic curve*	$\mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6$
elliptic curve	*2-torus*	$\mathfrak{h}_8$
Kodaira surface	*elliptic curve*	$\mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_{13}, \mathfrak{h}_{14}, \mathfrak{h}_{15}, \mathfrak{h}_{16}$

In particular the geometry is already determined by the real Lie algebra $\mathfrak{g}$. *Every deformation in the large of* M *has the same structure.*

If $\mathfrak{g} = \mathfrak{h}_7$ *then there is a dense subset of the space of all left-invariant complex structures for which* M *admits the structure of principal holomorphic bundle of elliptic curves over a Kodaira surface but this is not true for all complex structures.*

The remaining cases $\mathfrak{h}_{19}^-$ *and* $\mathfrak{h}_{26}^+$ *do not admit the structure of principal holomorphic torus bundle for any complex structure.*

In the case of $\mathfrak{h}_7$ the map underlying the fibration is not determined by the Lie algebra structure as we have already seen in example 1.14.

Proof. We prove our claims in reverse order.

The Lie algebras $\mathfrak{h}_{19}^-$ and $\mathfrak{h}_{26}^+$ do not admit any nilpotent complex structure (see [Sal01] or note that the centre has real dimension one) and therefore a corresponding nilmanifold can never admit an iterated principal holomorphic torus bundle structure.

A nilmanifold of type $(\mathfrak{h}_7, J, \Gamma)$ admits a structure of principal holomorphic torus bundle if and only if the subspaces $\mathcal{T}^1\mathfrak{h}_7$ and $\mathcal{T}^2\mathfrak{h}_7$ are Γ-rational. This is clearly the case if the complex structure is rational but not always as seen in 1.14. We have $\mathcal{T}^2\mathfrak{h}_7 \neq \mathfrak{h}_7$ since every quotient of $\mathfrak{h}_7$ by a two-dimensional subspace of the centre will be non abelian; hence we have a fibration of elliptic curves over a Kodaira surface if any. This is in fact the simplest example for Proposition 5.9 (*iv*).

It remains to treat the cases listed in the table. First of all note that every nilpotent Lie algebra of real dimension at most 4 which admits complex structures gives rise to a good fibre class, since the only possibilities are elliptic curves, 2-dimensional tori and Kodaira surfaces.

Hence we have to exhibit a stable torus bundle series with the appropriate dimensions for all Lie algebras given in the table: The Lie algebra $\mathfrak{h}_1$ corresponds to a 3-dimensional torus which has already been discussed. We list the the remaining cases together with the corresponding propositions from Section 5:

5.2	5.6 (*i*)	5.6 (*ii*)	5.6 (*iii*)	5.9 (*iii*) (b)
$\mathfrak{h}_3$, $\mathfrak{h}_8$	$\mathfrak{h}_9$	$\mathfrak{h}_6$	$\mathfrak{h}_2$, $\mathfrak{h}_4$, $\mathfrak{h}_5$	$\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$, $\mathfrak{h}_{13}$, $\mathfrak{h}_{14}$, $\mathfrak{h}_{15}$, $\mathfrak{h}_{16}$

Skipping the calculation of the of the various dimensions we conclude the proof. □

Remark 6.4 — The real Lie algebra underlying the Iwasawa manifold is isomorphic to $\mathfrak{h}_5$ and hence we have in particular proved that every deformation in the large of the Iwasawa manifold is a nilmanifold with left-invariant complex structure.

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