DOUBLE KODAIRA FIBRATIONS

FABRIZIO CATANESE AND SÖNKE ROLLENSKE

Abstract. The existence of a Kodaira fibration, i.e., of a fibration of a compact complex surface S onto a complex curve B which is a differentiable but not a holomorphic bundle, forces the geographical slope $\nu(S) = c_1^2(S)/c_2(S)$ to lie in the interval $(2, 3)$. But up to now all the known examples had slope $\nu(S) \leq 2 + 1/3$. In this paper we consider a special class of surfaces admitting two such Kodaira fibrations, and we can construct many new examples, showing in particular that there are such fibrations attaining the slope $\nu(S) = 2 + 2/3$. We are able to explicitly describe the moduli space of such class of surfaces, and we show the existence of Kodaira fibrations which yield rigid surfaces. We observe an interesting connection between the problem of the slope of Kodaira fibrations and a 'packing' problem for automorphisms of algebraic curves of genus ≥ 2.

AMS classification: 14J29, 14J25, 14D06, 14H37, 32Q55, 32G15.

Contents

1. Introduction.

It is well known that the topological Euler characteristic e is multiplicative for fibre bundles and in 1957 Chern, Hirzebruch and Serre ([CHS57]) showed that the same holds true for the signature σ if the fundamental group of the base acts trivially on the (rational) cohomology of the fibre.

In 1967 Kodaira [Kod67] constructed examples of fibrations of a complex algebraic surface over a curve which are differentiable fibre bundles for which the multiplicativity of the signature does not hold true. In his honour such fibrations are nowadays called Kodaira fibrations.

Definition 1.1 *A **Kodaira fibration** is a smooth holomorphic fibration $\psi : S \to B$ of a compact complex surface over a compact complex curve, which is not a (locally trivial) holomorphic fibre bundle.*

The hypothesis that ψ is a fibration means that every fibre F is connected, that ψ is smooth means that every fibre F is nonsingular. We denote by g the genus of F, respectively by b the genus of the base curve B.

Note that by [F-G65] ψ is a holomorphic fibre bundle if and only if all the fibres are isomorphic.

It is well known that one has a holomorphic bundle if the fibre genus g is ≤ 2, and there are no singular fibres. Likewise the genus b of the base curve of a Kodaira fibration has to be ≥ 2.

Atiyah and Hirzebruch ([At69], [Hirz69]) presented variants of Kodaira's construction analysing the relation of the monodromy action to the non multiplicativity of the signature.

Other constructions of Kodaira fibrations have been later given by Gonzalez-Diez and Harvey and others (see [GD-H91], [Zaal95], [B-D02] and references therein) in order to obtain fibrations over curves of small genus with fixed signature and fixed fibre genus.

A precise quantitative measure of the non-multiplicativity of the signature is given by the *geographic slope*, i.e., the ratio $\nu := c_1^2(S)/c_2(S) = K_S^2/e(S)$ between the Chern numbers of the surface: for Kodaira fibred surfaces it lies in the interval $(2, 3)$, in view of the well known Arakelov inequality and of the improvement by Kefeng Liu ([Liu96]) of the Bogomolov-Miyaoka-Yau inequality $K_S^2/e(S) \leq 3$.

The basic problem we approach in this paper is : which are the slopes of Kodaira fibrations?

This problem was posed by Claude Le Brun who raised the question whether the slopes can be bounded away from 3: is it true for instance that for a Kodaira fibration the slope is at most 2,91? In fact, the examples by Atiyah, Hirzebruch and Kodaira have slope not greater than $2+1/3 = 2,33\ldots$ (see [BHPV], page. 221) and if one considers Kodaira fibrations obtained from a general complete intersection curve in the moduli space $\mathfrak{M}_g$ of curves of genus $g \geq 3$, one obtains a smaller slope (around 2,18).

Our main result in this direction is the following

Theorem A *There are Kodaira fibrations with slope equal to* $2 + 2/3 = 2, 66\ldots$.

Our method of construction is a variant of the one used by Kodaira, and is briefly described as follows: we consider branched coverings $S \to B_1 \times B_2$ branched over a smooth divisor $D \subset B_1 \times B_2$ such that the respective projections $D \to B_i$ are étale (unramified) for $i = 1, 2$. We call these *double étale Kodaira fibrations*.

The advantage of this construction is that we are able to completely describe the moduli spaces of these surfaces.

The starting point is the topological characterization (derived from [Kot99]) of the surfaces which admit two different Kodaira fibrations, called here **double Kodaira fibred surfaces** .

Proposition 2.5 *Let S be a complex surface. A double Kodaira fibration on S is equivalent to the datum of two exact sequences*

$$1 \longrightarrow \Pi_{g_i} \longrightarrow \pi_1(S) \xrightarrow{\bar{\psi}_i} \Pi_{b_i} \longrightarrow 1 \qquad i = 1, 2\ ,$$

(here Π_g denotes the fundamental group of a compact curve of genus g) such that:

(*i*) $b_i \geq 2, g_i \geq 3$,

(*ii*) *the composition homomorphism*

$$\Pi_{g_1} \longrightarrow \pi_1(S) \xrightarrow{\bar{\psi}_2} \Pi_{b_2}$$

is neither zero nor injective, and

(*iii*) *the Euler characteristic of S satisfies*

$$e(S) = 4(b_1 - 1)(g_1 - 1) = 4(b_2 - 1)(g_2 - 1).$$

The above result shows that surfaces admitting a double Kodaira fibration form a closed and open subset in the moduli spaces of surfaces of general type; since for these one has a realization as a branched covering $S \to B_1 \times B_2$, branched over a divisor $D \subset B_1 \times B_2$, it makes sense to distinguish the étale case where D is smooth and the two projections $D \to B_i$ are étale. It is not clear a priori that this property is also open and closed, but we are able to prove it.

Theorem 6.4 *Double étale Kodaira fibrations form a closed and open subset in the moduli spaces of surfaces of general type.*

We can speak then of the moduli spaces of double étale Kodaira fibred surfaces S: they are given by a union of connected components of the moduli spaces of surfaces of general type. We conjecture that these connected components are irreducible, and we are able to prove this conjecture in the special case of *standard* double étale Kodaira fibred surfaces, the case which is most interesting since we have there lots of concrete examples.

To explain what we mean by the 'standard' case, the case where double étale Kodaira fibrations are constructed starting from curves with automorphisms, let us see how a double étale Kodaira fibration is related to a set of étale morphisms between two fixed curves.

In fact, every component of the branch divisor $D \subset B_1 \times B_2$ is an étale covering of each B_i, in particular of B_1. Hence we can take an étale cover $\pi : \tilde{B}_1 \to B_1$ dominating each of them; then the pullback, i.e., the fibre-product $S' := S \times_{B_1} \tilde{B}_1$, 'is a double étale Kodaira fibration' $S' \to \tilde{B}_1 \times B_2$ with the property that its branch divisor D' is composed of disjoint graphs of étale maps $\phi_i : \tilde{B}_1 \to B_2$.

The philosophy, as the reader may guess, is then: the larger the cardinality of $\mathcal{S} = \{\phi_i\}$ compared to the genus of B_2, the bigger the slope, and conversely, once we find such a set $\mathcal{S}$ we get (by the so-called tautological construction, described in section 4) plenty of corresponding double (étale) Kodaira fibrations. If by a further pullback we can achieve $B_1 = B_2 = B$ and $\mathcal{S} \subset \mathrm{Aut}(B)$, our question concerning the slope of double Kodaira fibrations is related to the following question.

Question B *Let B be a compact complex curve of genus $b \geq 2$, and let $\mathcal{S} \subset \mathrm{Aut}(B)$ be a subset such that all the graphs $\Gamma_s, s \in \mathcal{S}$ are disjoint in $B \times B$: which is the best upper bound for $|\mathcal{S}|/(b-1)$?*

We find examples with $|\mathcal{S}|/(b-1) = 3$, and in this way we obtain the slope $8/3$. Conversely, it is interesting to observe that the cited upper bound for the slope ($\nu(S) < 3$) implies that $|\mathcal{S}|/(b-1) < 8$.

It would be desirable to find examples with $|\mathcal{S}|/(b-1) > 3$, for instance examples with $|\mathcal{S}|/(b-1) = 4$ would yield a slope equal to $2,75$. Even more interesting would be to find sharper upper bounds for the slope of Kodaira fibrations.

The consideration of double étale Kodaira fibrations related to curves with many automorphisms enables us also to prove the following interesting

Corollary 6.6 *There are double Kodaira fibred surfaces S which are rigid.*

In the last section we interpret the above corollary as an existence result for rigid curves in the moduli stack of genus 7 curves.

The moduli space of some special Kodaira fibrations were described by Kas [Ks68] and Jost/Yau [J-Y83]; here, we prove the following general

Theorem 6.5 *The subset of the moduli space corresponding to* standard *double étale Kodaira fibred surfaces S (those admitting a pullback branched in a union of graphs of automorphisms), is a union of connected components which are irreducible, and indeed isomorphic to the moduli space of pairs (B, G), where B is a*

curve of genus b at least two and G is a group of biholomorphisms of B of a given topological type.

2. General set-up.

Definition 2.1 *A* Kodaira fibration *is a smooth fibration $\psi_1 : S \to B_1$ of a compact complex surface over a compact complex curve, which is not a holomorphic fibre bundle.*

S is called a double Kodaira fibred surface *if it admits a* double Kodaira fibration*, i.e., a surjective holomorphic map $\psi : S \to B_1 \times B_2$ yielding two Kodaira fibrations $\psi_i : S \to B_i$ $(i = 1, 2)$.*
Let $D \subset B_1 \times B_2$ be the branch divisor of ψ. If D is smooth and both projections $pr_{B_j}|_D : D \to B_j$ are étale we call $\psi : S \to B_1 \times B_2$ a double étale Kodaira fibration.

Remark 2.2 — Observe that S admits a double Kodaira fibration, if and only if S admits two distinct Kodaira fibrations. If S admits two Kodaira fibrations $\psi_i : S \to B_i, i = 1, 2$ then we can consider the product morphism $\psi := \psi_1 \times \psi_2 : S \to B_1 \times B_2$. Let R be the ramification divisor of ψ and D its branch locus. We observe here R and D are not necessarily smooth, as shown by the following local computation.

Take in fact a point P of the ramification divisor R: since ψ_1 is of maximal rank, there are local holomorphic coordinates (x, y) on S and t_i on B_i such that P corresponds to the origin and, $t_1(\psi_1(x, y)) = x$, and $t_2(\psi_2(0, 0)) = 0$.

Hence, since P is a ramification point, we may assume without loss of generality that $\psi_2(x, y) = x + f(x, y)$. The local equation of R is $\frac{\partial f}{\partial y}(x, y) = 0$, while the local equation of B is given by the resultant $Res_y\left(\frac{\partial f}{\partial y}(t_1, y), t_1 + f(t_1, y) - t_2\right)$.

For instance, if $f(x, y) = yx^2 - 1/4y^4$, R is singular at P, with equation $y^3 = x^2$, and D is singular at the image point, with equation $3^3 t_1^8 = 4^3 t_2^3$.

Observe finally that a surface S could admit three or more different Kodaira fibrations.

Remark 2.3 — Arnold Kas remarked in [Ks68] that, if $\phi : S \to B$ is a Kodaira fibration, then the genus of the base is at least two and the genus g of the fibre is at least three.

In the case of a double Kodaira fibration the genus of the fibre is easily seen to be at least four by Hurwitz' formula, since a fibre of ψ_i is a branched covering of a curve of genus at least two, namely the base curve of the other fibration.

In particular, S cannot contain rational or elliptic curves, since no such curve is contained in a fibre or admits a non-constant map to the base curve. Hence S is minimal and one sees, using the superadditivity of Kodaira dimension, that S is an algebraic surface of general type.

Lemma 2.4 *Let S be a surface admitting two different smooth fibrations $\psi_i : S \to B_i$ where $b_i :=$ genus $(B_i) \geq 2$ and where the fibre genus also satisfies $g_i \geq 2$. If e.g. ψ_1 is a holomorphic fibre bundle map, then S has an étale covering which is isomorphic to a product of curves, $S \to B_1 \times B_2$ is étale, and also ψ_2 is a holomorphic fibre bundle.*

Proof. Let F be a fibre of ψ_1. Since the genus of F is at least two, the automorphism group of F is finite. Hence we can pull back S by an étale map $f : \tilde{B}_1 \to B_1$

to obtain a trivial bundle resulting in the diagram

$$\begin{array}{ccccc} \tilde{B}_1 \times F & \xrightarrow{\phi} & S & \xrightarrow{\psi} & B_1 \times B_2 \\ \downarrow & & \downarrow{\scriptstyle \psi_1} & \swarrow & \\ \tilde{B}_1 & \xrightarrow{f} & B_1 & & \end{array}$$

where $\psi := \psi_1 \times \psi_2$.

The composition $\psi \circ \phi$ maps $\{\tilde{b}\} \times F$ to $\{f(\tilde{b})\} \times B_2$ and we infer from [Cat00], Rigidity-Lemma 3.8, that this map does not depend on $\tilde{b} \in \tilde{B}_1$. In other words, there exists a map $g : F \to B_2$ such that $\psi \circ \phi = f \times g$.

Now we pick $x \in F$ and set $g(x) := y$, $S_y := \psi_2^{-1}(y)$. In the diagram

$$\begin{array}{ccc} \tilde{B}_1 \times \{x\} & \xrightarrow{\phi} & S_y \\ & \searrow^{f} & \downarrow{\scriptstyle \psi|_{S_y}} \\ & & B_1 \times \{y\} \end{array}$$

ϕ and f are étale and consequently also $\psi|_{S_y}$ is étale. Since this holds $\forall x \in F$, whence $\forall y \in B_2$, we see that ψ and g are étale. Now any fibre of ψ_2 is an étale covering of B_1 of fixed degree, corresponding to a fixed subgroup of $\pi_1(B_1)$. Thus the fibres are all isomorphic and we have a holomorphic bundle (by [F-G65]). □

We can now give a topological characterization of double Kodaira fibrations. We denote by Π_g the fundamental group of a compact complex curve of genus g.

Proposition 2.5 *Let S be a complex surface. A double Kodaira fibration on S is equivalent to the datum of two exact sequences*

$$1 \longrightarrow \Pi_{g_i} \longrightarrow \pi_1(S) \xrightarrow{\bar{\psi}_i} \Pi_{b_i} \longrightarrow 1 \qquad i = 1, 2\ ,$$

(here Π_g denotes the fundamental group of a compact curve of genus g) such that:

(*i*) $b_i \geq 2, g_i \geq 3$,

(*ii*) *the composition homomorphism*

$$\Pi_{g_1} \longrightarrow \pi_1(S) \xrightarrow{\bar{\psi}_2} \Pi_{b_2}$$

is neither zero nor injective, and

(*iii*) *the Euler characteristic of S satisfies*

$$e(S) = 4(b_1 - 1)(g_1 - 1) = 4(b_2 - 1)(g_2 - 1).$$

Proof. Let us make a preliminary remark. Note that a holomorphic map $f : C' \to C$ between algebraic curves of genus at least 1 is étale if and only if the induced map f_* on the fundamental groups is injective. In fact, in this case there is a covering space $g : D \to C$ corresponding to the subgroup $f_*(\pi_1(C'))$ in $\pi_1(C)$ and by the lifting theorem we have a holomorphic map $\tilde{f} : C' \to D$ which induces an isomorphism of the fundamental groups. Hence C', D have the same genus and $\tilde{f}$ is an isomorphism by Hurwitz' s formula if the genus of D is ≥ 2, while if D has genus 1, then it is étale, whence an isomorphism since it induces an isomorphism of fundamental groups. The conclusion is that also $f = g \circ \tilde{f}$ is étale.

Given now a double Kodaira fibration, the above exact sequences are just the homotopy exact sequences of the two differentiable fibre bundles ψ_i. We observed already that (*i*) holds, while (*iii*) is the multiplicativity of the topological Euler number for fibre bundles (in algebraic geometry, it is called the Zeuthen-Segre

formula). Since the two fibrations are different, the map in (ii) cannot be zero. Furthermore, it cannot be injective by Lemma 2.4, hence (ii) holds.

Assume conversely that we have two exact sequences satisfying the above conditions (i), (ii) and (iii). Using theorem 6.3 of [Cat03] (i) and (iii) guarantee the existence of two curves B_i of genus b_i and of holomorphic submersions $\psi_i : S \to B_i$ with $\psi_{i*} = \bar{\psi}_i$ whose fibres have respective genera g_1, g_2.

Condition (ii) implies that the two fibrations are different and it remains to see that neither of the ψ_i's can be a holomorphic bundle. But if it were so, by Lemma 2.4, then $S \to B_1 \times B_2$ would be étale and the map in (ii) would be injective. □

Remark 2.6 — Double Kodaira fibrations which are not double étale were constructed in [GD-H91] and [Zaal95], essentially with the same method. The map $F : B \times B \to Jac(B)$, $(x, y) \mapsto x - y$ contracts the diagonal $\Delta_B \subset B \times B$ and maps $B \times B$ to $Y := B - B \subset Jac(B)$. One takes $\Gamma \subset Y$ to be a general very ample divisor, and $D \subset \Gamma \times B$ as $D := \cup_{x \in \Gamma} F^{-1}(x)$. The projection of D to Γ is étale of degree 2, while the projection of D to B is of degree equal to $b := genus(B)$ but is not étale. The pair $D \subset \Gamma \times B$ yields, as we shall explain in a forthcoming section, a 'logarithmic Kodaira fibration', and from it one can construct, via the tautological construction, an actual Kodaira fibration.

We shall be primarily interested in the case of double étale Kodaira fibrations. Given a holomorphic map ϕ between two curves let us denote by Γ_ϕ its graph.

Definition 2.7 *A double étale Kodaira fibration $S \to B_1 \times B_2$ is said to be* simple *if there exist étale maps $\phi_1, \dots, \phi_m$ from B_1 to B_2 such that $D = \bigcup_{k=1}^{m} \Gamma_{\phi_k}$; i.e., if each component of D is the graph of one of the ϕ_k's.*

We say that S is very simple *if $B_1 = B_2$ and all the ϕ_k's are automorphisms.*

Lemma 2.8 *Every double étale Kodaira fibration admits an étale pullback which is simple.*

Proof. Let $S \to B_1 \times B_2$ be a double étale Kodaira fibration. The branch divisor D is smooth and we can consider the monodromy map $\mu : \pi_1(B_1, b_1) \to S_{m_1}$ of the étale map $p_1 : D \to B_1$. Let $f : B \to B_1$ the (finite) covering associated to the kernel of μ_1. By construction the monodromy of the pullback $f^*D \to B$ is trivial, hence every component maps to B with degree 1 and the corresponding pullback $f^*S := B \times_{B_1} S$ is a simple Kodaira fibration. □

Remark 2.9 — Kollár claimed that it should be possible to construct double étale Kodaira fibrations which do not admit any étale pullback which is very simple. But up to now we do not know any example of this situation.

This motivates the following:

Definition 2.10 *A double étale Kodaira fibration is called* **standard** *if there exist étale Galois covers $B \to B_i, i = 1, 2$, such that the étale pullback*

$$S' := S \times_{(B_1 \times B_2)} (B \times B),$$

induced by $B \times B \to B_1 \times B_2$, is very simple.

3. Invariants of double étale Kodaira fibrations

In this section we want to calculate some invariants of a double étale Kodaira fibration. First we need to fix the notation.

Let S be a double étale Kodaira fibration as in Definition 2.1. Let d be the degree of $\psi : S \to B_1 \times B_2$, let $D \subset B_1 \times B_2$ be the branch locus of ψ and let $D_1, \dots, D_m$ be the connected components of D.

By assumption, the composition map

$$D_i \hookrightarrow B_1 \times B_2 \stackrel{pr_j}{\to} B_j$$

is étale and we denote by d_{ij} its degree. Then the degree of $pr_j|_D : D \to B_j$ is $d_j = \sum_{i=1}^m d_{ij}$ and we get two formulae for the Euler characteristic of D_i,

$$e(D_i) = d_{i1}e(B_1) = d_{i2}e(B_2).$$

The canonical divisor $K_{B_1\times B_2}$ of $B_1 \times B_2$ is numerically (indeed, algebraically) equivalent to $-e(B_1)B_2 - e(B_2)B_1$, where we denote by B_1 any divisor of the form $B_1 \times \{y\}$ (similarly for B_2), and we calculate

$$\begin{aligned} K_{B_1\times B_2} \cdot D_i &= -e(B_1)B_2 \cdot D_i - e(B_2)B_1 \cdot D_i \\ &= -e(B_1)d_{i1} - e(B_2)d_{i2} = -2e(D_i) \end{aligned}$$

so that by adjunction

$$\begin{aligned} D_i^2 &= deg(K_{D_i}) - K_{B_1\times B_2} \cdot D_i \\ &= -e(D_i) + 2e(D_i) = e(D_i). \end{aligned}$$

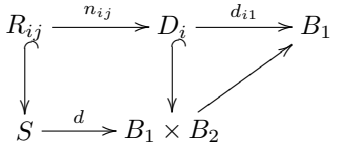
We write

$$\psi^{-1}(D_i) = \bigcup_{j=1}^{t_i} R_{ij}$$

as a union of disjoint curves and denote by n_{ij} the degree of $\psi|_{R_{ij}} : R_{ij} \to D_i$ and by r_{ij} the branching order of ψ along R_{ij} (i.e., the multiplicity of the reduced divisor R_{ij} in the full transform of D_i). Then

$$K_S = \psi^* K_{B_1\times B_2} + \sum_{i,j}(r_{ij} - 1)R_{ij} \text{ and } d = \sum_{j=1}^{t_i} n_{ij}r_{ij}.$$

To summarize the situation we label the arrows in the following diagram by the degrees of the corresponding maps:



Proposition 3.1 *In the above situation we have the following formulae*

(*i*) *Setting* $\beta_i := \sum_{j=1}^{t_i} n_{ij}(r_{ij} - 1)$

$$c_2(S) = d\, c_2(B_1 \times B_2) - \sum_{i=1}^m \beta_i e(D_i)$$

$$c_1^2(S) = 2c_2(S) - \sum_{i=1}^m \sum_{j=1}^{t_i} \frac{n_{ij}(r_{ij} - 1)(r_{ij} + 1)}{r_{ij}} e(D_i)$$

thus the signature is

$$\sigma(S) = \frac{1}{3}(c_1^2(S) - 2c_2(S)) = -\frac{1}{3}\sum_{i=1}^m \sum_{j=1}^{t_i} \frac{n_{ij}(r_{ij} - 1)(r_{ij} + 1)}{r_{ij}} e(D_i)$$

(*ii*) (a) *If* $\psi : S \to B_1 \times B_2$ *is a Galois covering then* $r_{ij} = r_i$ *and*

$$\frac{c_1^2(S)}{c_2(S)} = 2 + \frac{-\sum_{i=1}^m \frac{r_i^2-1}{r_i^2} e(D_i)}{e(B_1)e(B_2) - \sum_{i=1}^m \frac{r_i-1}{r_i} e(D_i)}.$$

(b) *If in addition D is composed of graphs of étale maps from B_1 to B_2, i.e., S is simple, we have*

$$\frac{c_1^2(S)}{c_2(S)} = 2 + \frac{1 - \frac{1}{m}\sum_{i=1}^m \frac{1}{r_i^2}}{\frac{2g-2}{m} + 1 - \frac{1}{m}\sum_{i=1}^m \frac{1}{r_i}}$$

where g is the genus of B_2.

Proof. The first formula can be obtained by calculating the genus of a fibre F of $S \to B_1$ using the Riemann-Hurwitz formula and using $c_2(S) = e(S) = e(B_1)e(F)$.

For the second one a rather tedious calculation of intersection numbers is needed so that we prefer to cite [Iz03][1] which gives us

$$\begin{aligned} c_1^2(S) &= d\, c_1^2(B_1 \times B_2) - \sum_{i=1}^m \left(2\ b_i\ e(D_i) + \sum_{j=1}^{t_i} \frac{n_{ij}(r_{ij}-1)(r_{ij}+1)}{r_{ij}} D_i^2 \right) \\ &= 2d\ e(B_1)e(B_2) - \sum_{i=1}^m 2b_i\ e(D_i) - \sum_{i=1}^m \sum_{j=1}^{t_i} \frac{n_{ij}(r_{ij}-1)(r_{ij}+1)}{r_{ij}} e(D_i) \\ &= 2c_2(S) - \sum_{i=1}^m \sum_{j=1}^{t_i} \frac{n_{ij}(r_{ij}-1)(r_{ij}+1)}{r_{ij}} e(D_i). \end{aligned}$$

The formula for the signature is now obvious.

Let's look at (ii). If the covering ψ is Galois the stabilizers of R_{ik} and R_{il} are conjugate in the covering group and consequently $n_{ij} = n_i$ and $r_{ij} = r_i$ do not depend on j. Hence for every i

$$d = t_i n_i r_i \Leftrightarrow \frac{t_i n_i}{d} = \frac{1}{r_i} \text{ and } \beta_i = t_i n_i (r_i - 1) = d - t_i n_i$$

Plugging this into the above formulae we get (a):

$$\begin{aligned} \frac{c_1^2(S)}{c_2(S)} - 2 &= \frac{-\sum_{i=1}^m \frac{t_i n_i (r_i-1)(r_i+1)}{r_i} e(D_i)}{d e(B_1 \times B_2) - \sum_{i=1}^m \beta_i e(D_i)} \\ &= \frac{-\sum_{i=1}^m \frac{r_i t_i n_i (r_i-1)(r_i+1)}{r_i^2} e(D_i)}{d\left(e(B_1 \times B_2) - \frac{1}{d}\sum_{i=1}^m (d - t_i n_i) e(D_i)\right)} \\ &= \frac{-\sum_{i=1}^m \frac{r_i^2-1}{r_i^2} e(D_i)}{e(B_1)e(B_2) - \sum_{i=1}^m \frac{r_i - 1}{r_i} e(D_i)} \end{aligned}$$

For (b) we further assume the components of D to have all the same genus as B_2, i.e. $e(D_i) = e(B_2)$ for all i. Then

$$\begin{aligned} \frac{c_1^2(S)}{c_2(S)} - 2 &= \frac{-\sum_{i=1}^m \frac{r_i^2-1}{r_i^2} e(B_2)}{e(B_1)e(B_2) - \sum_{i=1}^m \frac{r_i-1}{r_i} e(B_2)} \\ &= \frac{m - \sum_{i=1}^m \frac{1}{r_i^2}}{-e(B_1) + m - \sum_{i=1}^m \frac{1}{r_i}}. \\ &= \frac{1 - \frac{1}{m}\sum_{i=1}^m \frac{1}{r_i^2}}{\frac{2g-2}{m} + 1 - \frac{1}{m}\sum_{i=1}^m \frac{1}{r_i}} \end{aligned}$$

□

[1] Note that we have a slightly different notation.

The above formulae will allow us to give upper bounds for the slope of double étale Kodaira fibrations under some conditions.

4. Tautological construction

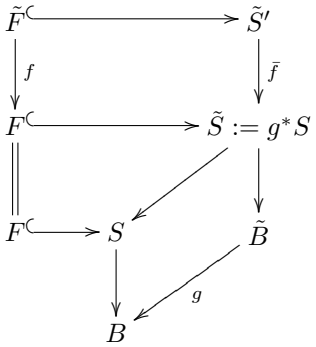
Definition 4.1 *A* **log-Kodaira fibration** *is a pair (S, D) consisting of*

(*i*) *a smooth fibration $\psi : S \to B$ with fibres F_t and*
(*ii*) *a divisor $D \subset S$ such that*
  (a) *the projection $D \to B$ is étale and*
  (b) *the fibration of pointed curves $(F_t, F_t \setminus D)$ is not isotrivial, i.e., the fibres are not all isomorphic (as pointed curves).*

Our typical example of the above situation will be a product of curves $S := B_1 \times B_2$ together with a divisor D such that the first projection $D \to B_1$ is étale and the second projection $D \to B_2$ is finite.

We shall now see that in order to construct Kodaira fibrations it suffices to construct log-Kodaira fibrations.

Proposition 4.2 *Let $(S, D) \to B$ be a log-Kodaira fibration and let $f : \tilde{F} \to F$ be a Galois-covering of a fibre F, with Galois group G, and branched over $D \cap F$. Then we can extend f to a ramified covering of surfaces $\bar{f} : \tilde{S}' \to \tilde{S}$ obtaining a diagram*



where

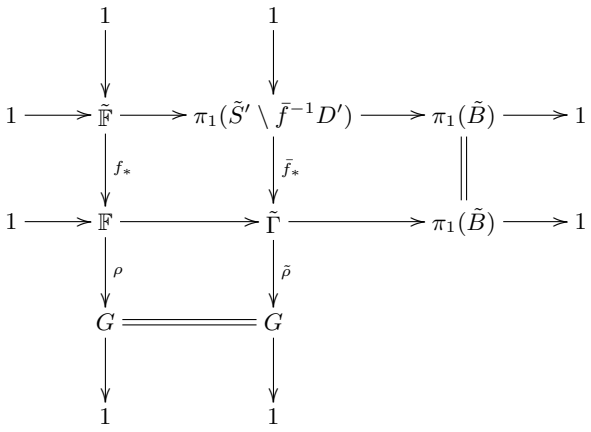
- *$g : \tilde{B} \to B$ is an étale covering,*
- *$\tilde{S}$ is the pullback of S via g,*
- *$\bar{f}$ is a ramified covering with Galois group G branched over $\tilde{D} := g^*D$ and such that $\bar{f}_{|\tilde{F}} = f$.*

Proof. First we translate the problem into a group-theoretical question by looking at the above desired situation in terms of fundamental groups.

Set for convenience $\hat{F} := F \setminus D$, $\hat{S} := S \setminus D$, $\mathbb{F} := \pi_1(\hat{F})$, $\Gamma := \pi_1(\hat{S})$ and $\tilde{\Gamma} = \pi_1(\tilde{S} \setminus \tilde{D})$. Then the fibre bundle $\hat{S} \to B$ and its étale pullback via g give rise to a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{F} & \longrightarrow & \tilde{\Gamma} & \longrightarrow & \pi_1(\tilde{B}) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle g_*} & & \\ 1 & \longrightarrow & \mathbb{F} & \longrightarrow & \Gamma & \longrightarrow & \pi_1(B) & \longrightarrow & 1. \end{array}$$

The ramified coverings of F and $\tilde{S}$ with Galois group G then would yield a diagram of exact sequences of group homomorphisms:



In other words, in order to prove the proposition we have to find a subgroup of finite index $\tilde{\Gamma} < \Gamma$ such that $\mathbb{F} \lhd \tilde{\Gamma}$ and such that the homomorphism $\rho : \mathbb{F} \to G$ extends to a homomorphism $\tilde{\rho} : \tilde{\Gamma} \to G$.

We will need several étale pullbacks in order to

- (1) make the pullback D' of the divisor D a union of sections,
- (2) kill the monodromy which prevents us from extending the homomorphism,
- (3) make the selfintersection of each component of D' divisible by the exponent of the group G (the minimal integer k such that $g^k = 1 \; \forall g \in G$).

Step (1) is achieved by induction on the intersection number $D \cdot F$, since

(i) the pull-back of a section is always a section, while

(ii) if D_1 is a component of D which is not a section, the pull-back of D under the covering $D_1 \to B$ contains a new section (namely, the diagonal of $D_1 \times D_1 \subset D_1 \times_B S$).

For step (2), since $\mathbb{F}$ is a normal subgroup, $\gamma \in \Gamma$ operates on $\mathbb{F}$ by conjugation and hence on $\mathrm{Hom}(\mathbb{F}, G)$ via $\phi \mapsto \gamma(\phi) = \phi \circ \mathrm{Int}_{\gamma^{-1}}$.

Let Γ_ρ be the stabilizer of ρ under this action. For $\gamma \in \Gamma_\rho$ it holds $\rho(\gamma x \gamma^{-1}) = \rho(x)$ and in particular γ normalizes $\tilde{\mathbb{F}}$, the kernel of ρ (this should certainly be true for elements in $\tilde{\Gamma}$). Let Γ' be the subgroup of Γ generated by $\mathbb{F}$ and Γ_ρ.

Note that since $\mathbb{F}$ is normal in Γ we can write every element $\gamma' \in \Gamma'$ as a product $\gamma' = fg$ where $f \in \mathbb{F}$ and $g \in \Gamma_\rho$.

The subgroup $\Gamma' < \Gamma$ gives rise to an exact sequence

$$1 \to \mathbb{F} \to \Gamma' \to \Pi' \to 1$$

(where Π' is a subgroup of finite index of $\pi(B)$), hence to an étale pullback $\pi : S' \to S$.

Let D_0 be a component of $\pi^{-1}D$, the pullback of the branch divisor, and let N_0 be a tubular neighbourhood of D_0. Let γ_0 be a small loop around D_0 contained in $N_0 \cap F$. We consider γ_0 also as an element of $\mathbb{F}$ and regard N_0 as a small neighbourhood of the zero section in the normal bundle $\mathcal{N}_{D_0/S}$.

The idea is to extend the homomorphism ρ to the surface by moving along D_0.

Observe that, since D_0 is a section, the fundamental group $\pi_1(D_0)$ is equal to Π'.

Lemma 4.3 *Let* $\Pi' := \pi_1(D_0) = \langle \alpha_i, \beta_i \mid \prod_i [\alpha_i, \beta_i] = 1 \rangle$. *Then* $\Gamma'' := \pi_1(N_0 \setminus D_0)$ *is a central extension*

$$1 \to \mathbb{Z} \to \Gamma'' \to \Pi' \to 1$$

where $\prod_i [\alpha_i, \beta_i] = \gamma_0^k$ *and* $k = -D_0^2$.

Proof. (See [Cat06], page 139) Pick a point $P \in D_0$ and write $D_0 = (D_0 \setminus P) \cup \Delta_P$ where Δ_P is a small disk around P. The S^1 bundle (homotopically equivalent to) $N_0 \setminus D_0$ restricted to these two open subsets is trivial and the $\mathcal{C}^\infty$ -cocycle of N_0 with regard to this trivialisation can be given as z^k, where z is a local coordinate in P and $k = -c_1(\mathcal{N}_{D_0/S}) = -D_0^2$. The fundamental group of N_0 is then calculated using the Seifert-van Kampen theorem. □

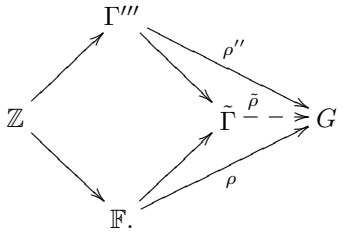
By a further base change we may assume that γ_0^k is in $\tilde{\mathbb{F}}$ (i.e., $\rho(\gamma_0)^k = 1$). In fact, since D_0 is a section, its pull-back under an étale base change (with base that we shall denote by $\tilde{B}$) of order d yields a new section D_0' whose selfintersection $-{D_0'}^2 = -dD_0^2$, and it suffices to choose d divisible by the the order of G (the exponent of G indeed suffices).

Defining Π'' as the fundamental group $\pi_1(\tilde{B})$ of the base curve obtained by the above procedure, and $\tilde{\Gamma}$ as the inverse image of Π'' in Γ, we obtain a new diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma''' & \longrightarrow & \Pi'' & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{F} & \longrightarrow & \tilde{\Gamma} & \longrightarrow & \Pi'' & \longrightarrow & 1 \end{array}$$

and may finally extend ρ to $\tilde{\Gamma}$ as follows: after choosing arbitrary images $\rho''(\alpha_i) = \rho''(\beta_i)$ in the centre of G and setting $\rho''(\gamma_0) := \rho(\gamma_0)$, we get a homomorphism $\rho'' : \Gamma'' \to G$ which coincides with ρ on γ_0 (Every such assignment is compatible with the relations in the group).

We are now in the situation



Writing each $\tilde{\gamma} \in \tilde{\Gamma}$ as a product $\tilde{\gamma} = fg$ with $f \in \mathbb{F}$ and $g \in \Gamma'''$ we define

$$\tilde{\rho}(\tilde{\gamma}) := \rho(f) \cdot \rho''(g).$$

To see that this is well defined let $f_1 g_1 = f_2 g_2$. Then ${f_2}^{-1} f_1 = g_2 {g_1}^{-1}$ is an element of $\mathbb{F}$ and of Γ''', i.e., a multiple of γ_0. Hence, applying ρ and ρ'' respectively, we get $\rho(f_2)^{-1}\rho(f_1) = \rho''(g_2)\rho''(g_1)^{-1}$ since the two homomorphisms act in the same way on γ_0. This implies $\rho(f_1 g_1) = \rho(f_2 g_2)$.

It remains to check that this defines a homomorphism. We consider two elements $f_1 g_1$, $f_2 g_2$ as above. Since Γ''' is contained in Γ' we can actually assume that the g_i's can be written as a combination of the α_i's, β_i's and are contained in Γ_ρ, hence

they stabilize ρ. Now

$$\begin{aligned}\rho(f_1g_1f_2g_2) &= \rho(f_1g_1f_2({g_1}^{-1}g_1)g_2)\\ &= \rho(f_1g_1f_2{g_1}^{-1})\rho''(g_1g_2)\\ &= \rho(f_1)\rho(g_1f_2{g_1}^{-1})\rho''(g_1)\rho''(g_2)\\ &= \rho(f_1)\rho(f_2)\rho''(g_1)\rho''(g_2)\\ &= \rho(f_1)\rho''(g_1)\rho(f_2)\rho''(g_2) \qquad\qquad = \rho(f_1g_1)\rho(f_2g_2)\end{aligned}$$

where the last line follows because we have chosen the images of the α_i's, β_i's in the centre of G.

Hence we have extended ρ to a finite index subgroup $\tilde{\Gamma}$ of Γ and the Kodaira fibration $\tilde{S}'$ arises as the ramified Galois cover of $\tilde{S}$ associated to this homomorphism. □

If S is a double étale Kodaira fibration or a product of curves, one can easily see that also S' is a double étale Kodaira fibration, provided that the restriction of the second projection to D is étale. Moreover the following holds:

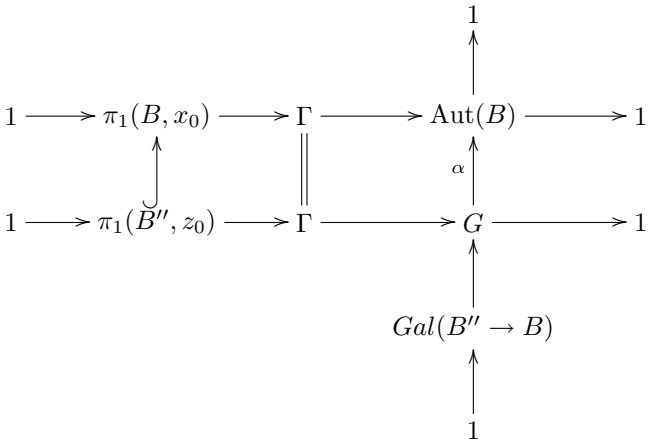
Lemma 4.4 *Assume that we have a curve B of genus at least two and a subset $\mathcal{S} = \{\phi_1, \dots, \phi_m\} \subset \operatorname{Aut} B$ such that the graphs of these automorphisms are disjoint subsets of $B \times B$. If we construct a Kodaira fibration applying the tautological construction to this log-Kodaira fibration, then the resulting surface is in fact a standard Kodaira fibration.*

Proof. Without loss of generality we may assume that $\phi_1 = id_B$, i.e., we identify the vertical and the horizontal part of the product $B \times B$ via the automorphism ϕ_1. We fix a base point x_0 in B. It suffices to prove the following: for any étale Galois covering $B' \to B$ there exists another étale covering map $f : B'' \to B' \to B$ such that the pullback of $D := \Gamma_{\phi_1} \cup \dots \cup \Gamma_{\phi_m}$ under the map $f \times f : B'' \times B'' \to B \times B$ is composed of graphs of automorphisms of B''.

The fundamental group $\pi_1(B, x_0)$ can be considered as a subgroup of a Fuchsian group which acts on the upper half plane. Let Γ be the maximal Fuchsian group which contains $\pi_1(B, x_0)$ as a normal subgroup. Then we have an exact sequence

$$1 \to \pi_1(B, x_0) \to \Gamma \to \operatorname{Aut}(B) \to 1.$$

The Galois covering $B' \to B$ corresponds to an inclusion $\pi_1(B', y_0) \subset \pi_1(B, x_0)$ where y_0 maps to x_0. Consider the Galois covering $B'' \to B' \to B$ associated to the subgroup $\pi(B'', z_0) := \bigcap_{\gamma \in \Gamma} \gamma\pi_1(B', y_0)\gamma^{-1}$ which is the largest normal subgroup of Γ contained in $\pi_1(B', y_0)$. It is in fact a finite index subgroup of $\pi_1(B', y_0)$ since $\pi_1(B', y_0)$ is of finite index in Γ. We have exact sequences



where G is a group of automorphisms of B''.

Let d be the degree of the covering $f : B'' \to B$. Then the degree of the map $f \times f : B'' \times B'' \to B \times B$ is d^2 and it suffices to exhibit for any given automorphism ϕ of B a set of d automorphisms of B'' such that their graphs map d to 1 to Γ_ϕ under the map $f \times f$. In order to do so pick $\psi \in G$ such that $\alpha(\psi) = \phi$ which means $f \circ \psi = \phi \circ f$. Then for any $\sigma \in Gal(B'' \to B)$ we have

$$\begin{aligned}(f \times f)(\Gamma_{\sigma\circ\psi}) &= (f \times f)(\{(x, y) \in B'' \times B'' \mid y = \sigma \circ \psi(x)\}) \\ &= \{(f(x), f(\sigma \circ \psi(x))) \mid x \in B''\} \\ &= \{(f(x), f \circ \psi(x)) \mid x \in B''\} \\ &= \{(f(x), \phi(f(x))) \mid x \in B''\} = \Gamma_\phi\end{aligned}$$

and this map has in fact the same degree as f. □

The reason why the monodromy problems mentioned in 2.9 do not occur in this case is that the horizontal and the vertical curve in the product are in fact identified via ϕ_1 and therefore, once we fix a basepoint on the curve during the tautological contruction, there is no ambiguity in the choice of the basepoint on the other curve.

5. Slope of double étale Kodaira fibrations

Kefeng Liu proved in [Liu96] that the slope ν of a Kodaira fibration S satisfies

$$\nu := \frac{c_1^2(S)}{c_2(S)} < 3$$

and Le Brun asked whether the better bound $c_1^2(S) < 2.91 c_2(S)$ would hold.

We will now address the question about what can be said for double étale Kodaira fibrations. Our purpose here is twofold: to find effective estimates from below for the maximal slope via the construction of explicit examples and then to see whether one can prove also an upper bound for the slope of double Kodaira fibrations, using their explicit description.

To separate the numerical considerations from the geometrical problems we pose the following

Definition 5.1 *Let B_1, B_2 be curves of genus at least two. An admissible configuration for $B_1 \times B_2$ is a tuple $\mathcal{A} = (D, d, \{(t_i, \{r_{ij}, n_{ij}\}\})$ consisting of*

- *a smooth curve $D = D_1 \cup \cdots \cup D_m \subset B_1 \times B_2$ such that each connected component D_i maps in an étale fashion to each of the factors,*
- *a positive integer d, and positive integers t_i, for all $i = 1, \ldots, m$,*
- *for all $i = 1, \ldots, m$, a t_i-tuple $\{(r_{ij}, n_{ij})\}_{j=1,\ldots,t_i}$ of pairs of positive integers with $r_{ij} \geq 2$, and such that*

$$d = \sum_{j=1}^{t_i} n_{ij} r_{ij}.$$

We call the configuration Galois if r_{ij} and n_{ij} do not depend on j, and we then write $\mathcal{A} = (D, d, \{(t_i, r_i, n_i)\})$. If moreover D is consists of graphs of étale maps $\phi_k : B_1 \to B_2$ (automorphisms if $B_1 \cong B_2$) we call $\mathcal{A}$ simple (resp.: very simple). Setting $\beta_i := \sum_{j=1}^{t_i} n_{ij}(r_{ij} - 1)$ we define the abstract slope of $\mathcal{A}$ by

$$\mathfrak{a}(\mathcal{A}) = 2 + \frac{-\sum_{i=1}^{m} \sum_{j=1}^{t_i} \frac{n_{ij}(r_{ij}-1)(r_{ij}+1)}{r_{ij}} e(D_i)}{d\, e(B_1 \times B_2) - \sum_{i=1}^{m} \beta_i \; e(D_i)}.$$

We have seen in section 3 that a double étale Kodaira fibration S gives rise to an admissible configuration $\mathcal{A}(S)$. If $\mathcal{A}$ is any admissible configuration and S a double étale Kodaira fibration with $\mathcal{A}(S) = \mathcal{A}$ we say that S realizes $\mathcal{A}$. In this case the abstract slope $\mathfrak{a}(\mathcal{A})$ coincides with the slope of S by Proposition 3.1. Note that we also calculated formulae for the abstract slope of (very) simple Galois configurations.

To attain a bound from above for the slope we can now independently study the questions

- Which is the maximal possible abstract slope for an admissible configuration?
- How to realize a given configuration?

We already addressed the second problem in section 4 and we shall proceed by analysing the case of very simple configurations.

5.1. **Packings of graphs of automorphisms.** In this section we let B be a curve of genus g and $G = \mathrm{Aut}(B)$ its automorphism group.We want to study sub*sets* of G such that the corresponding graphs do not intersect. We can translate this into a group-theoretical condition:

Lemma 5.2 *Let $P_1, \dots, P_n$ be the points in B which have a non trivial stabilizer $\Sigma_{P_i} < G$. Let $\pi_i : G \to G/\Sigma_{P_i}$ be the map that sends $\phi \in G$ to the left coset $\phi\Sigma_{P_i}$.*

(*i*) *Two automorphisms $\phi \neq \phi' \in G$ have intersecting graphs if and only if $\pi_i(\phi) = \pi_i(\phi')$ for some $i \in \{1, \dots, n\}$.*

(*ii*) *A subset $\mathcal{S} \subset G$ of cardinality m has non-intersecting graphs if and only if for each $i \in \{1, \dots, n\}$ the image of $\mathcal{S}$ under the map*

$$\pi_i : G \to G/\Sigma_{P_i} \qquad g \mapsto g\Sigma_{P_i}$$

has cardinality m. In particular:

$$m \leq \mathrm{Minimum}\{|G/\Sigma_{P_i}|\}_{i=1,\dots,n} \ .$$

Proof. To prove the first claim let ϕ, ϕ' be two automorphisms of B. Their graphs intersect in some point $(P_1, P_2) \in B \times B$ iff $\phi(P_1) = \phi'(P_1) = P_2$. But this means $\phi^{-1} \circ \phi'(P_1) = P_1$, i.e., $\phi^{-1} \circ \phi' \in \Sigma_{P_1}$ or equivalently $\phi\Sigma_{P_1} = \phi'\Sigma_{P_1}$.

The second claim is now an easy consequence. □

Note that, if $Q_1, \dots, Q_r$ are the branch points of the quotient map $B \to B/G$ and P_i' $(\forall i = 1, \dots r)$ is an arbitrary point in the inverse image of Q_i , then the non trivial stabilizers of points are exactly all the subgroups conjugated to the stabilizers $\Sigma_{P_i'}$ $(i = 1, \dots, r)$.

It is now a natural question to ask for the maximal possible m that one can realize, given a curve B, or given a fixed genus b (of B).

For the formulation of a partial result we introduce the following notation: we say that B is of type $(\nu_1, \dots, \nu_k)$ if B/G has genus zero and the map $B \to B/G$ is a ramified covering, branched over k points with respective multiplicities ν_i.

We always order the branch points so that $\nu_1 \leq \cdots \leq \nu_k$.

Proposition 5.3 (*i*) *If the genus g of B is at least two, the maximal cardinality m of a subset with non-intersecting graphs is smaller or equal to*

$3(g-1)$ *unless the type of* B *occurs in the following table:*

type	*upper bound for* m	$\lvert G\rvert$
(2,2,2,3)	$4(g-1)$	$12(g-1)$
(2,3,7)	$12(g-1)$	$84(g-1)$
(2,3,8)	$6(g-1)$	$48(g-1)$
(2,3,9)	$4(g-1)$	$36(g-1)$
(2,4,5)	$8(g-1)$	$40(g-1)$
(2,4,6)	$4(g-1)$	$24(g-1)$
(2,5,5)	$4(g-1)$	$20(g-1)$
(3,3,4)	$6(g-1)$	$24(g-1)$

(*ii*) *If the genus of* B *is small we get the following list:*

type	*upper bound for* m	*up to genus*
(2,2,2,3)	$2(g-1)$	30
(2,3,7)	$3(g-1)$	23
(2,3,8)	$3(g-1)$	23
(2,3,9)	$2(g-1)$	23
(2,4,5)	$2(g-1)$	23
(2,4,6)	$2(g-1)$	50
(2,5,5)	$4/3(g-1)$	50
(3,3,4)	$3(g-1)$	50

If the genus of the curve is one, we can clearly produce an arbitrarily large number of automorphisms with pairwise disjoint graphs by choosing appropriate translations.

Proof. Part (i) is a case by case analysis using the previous Lemma. Let B be a curve of genus $g \geq 2$, let G be its automorphism group and let h be the genus of B/G. Let $P_1, \ldots, P_k \in B/G$ be the branch points and $\nu_1 \leq \cdots \leq \nu_k$ be the corresponding indices (branching multiplicities). Then we have the Hurwitz formula

$$2g-2 = |G|\left(2h-2+\sum_{i=1}^{k}(1-\frac{1}{\nu_i})\right)$$

and by the lemma a maximal subset as above has at most cardinality

$$\mu := \frac{|G|}{\nu_k} = \frac{2g-2}{\nu_k\left(2h-2+\sum_{i=1}^{k}(1-\frac{1}{\nu_i})\right)},$$

where we set $\nu_1 = 1$ if there is no ramification. Note that the denominator can never be zero since this would imply $g = 1$. We distinguish the following cases:

$h \geq 2$: Clearly

$$\mu \leq \frac{2g-2}{\nu_k\left(2+\sum_{i=1}^{k}(1-\frac{1}{\nu_i})\right)} \leq g-1.$$

$h = 1$: We have

$$\mu \leq \frac{2g-2}{\nu_k\sum_{i=1}^{k}(1-\frac{1}{\nu_i})} \leq 2(g-1)$$

$h = 0$: Also in this case we necessarily have ramification and

$$\mu \leq \frac{2g-2}{\nu_k\left(-2+\sum_{i=1}^{k}(1-\frac{1}{\nu_i})\right)},$$

hence we have to check in which cases holds

$$0 < \lambda := \nu_k\left(-2+\sum_{i=1}^{k}(1-\frac{1}{\nu_i})\right) < \frac{2}{3}$$

Since $k \geq 5$ implies $\lambda \geq 1$ we have k at most 4, and $\nu_k > 2$. If $k = 4$ then $\lambda \geq (1/2)\nu_k - 1$ thus $\nu_k = 3$ and one sees immediately that $(2, 2, 2, 3)$ is the only possibility. If $k = 3$ one can check that $1 - \sum_{i=1}^{3} 1/\nu_i \geq 1 - 1/2 - 1/3 - 1/7 = 1/42$, (which corresponds to $|G| = 84(g-1)$), hence there are only finitely many cases for ν_k which are easy to consider and which yield exactly the remaining cases in the above table.

For part (ii) note that a finite group G can occur as an automorphism group of a curve of type $(\nu_1, \ldots, \nu_k)$ iff there are distinct elements $g_1, \ldots, g_k$ in G such that $g_1, \ldots, g_{k-1}$ generate G, $\prod_{i=1}^{k} g_i = 1$ and the order of g_j is ν_j. (cf. section 5.2 for a construction.) For all possible combinations of groups and generators up to the given genus, maximal subsets satisfying the conditions of the above Lemma were calculated using the program GAP and its database of groups of small order (cf. [GAP04]). □

Remark 5.4 — The bounds in the second table are sharp, that is, there exist examples that realize the given upper bound. The smallest group realizing $3(g-1)$ is $\mathbf{Sl}(2, \mathbb{Z}/3\mathbb{Z})$ acting on a curve of genus 2 of type $(3, 3, 4)$.

We will see in Remark 5.10 that the slope inequality obtained by Liu implies in fact the better bound $m < 8(g-1)$.

Question 5.5 — It is clear that we can realize the bound $m = 3(g-1)$ for arbitrary large genera g by taking Galois étale coverings of the examples we have obtained.

Can one prove that $3(g-1)$ is an upper bound for all curves?

5.2. **Bounds for the slope.** Since the slope of a Kodaira fibration does not change under étale pullback, by Lemma 2.8 it suffices to treat the slope for a simple configuration. We do this here for the Galois case.

Proposition 5.6 *Let $\mathcal{A} = (D_1 \cup \cdots \cup D_m, d, \{t_i, r_i, n_i\})$ be a simple, Galois configuration and let g be the genus of the target curve B_2. If $m \leq 3(g-1)$ then $\mathfrak{a}(\mathcal{A}) \leq 2 + 2/3$ with equality if and only if $m = 3(g-1)$ and all the branching multiplicities r_i are equal to three.*

Remark 5.7 — i) We believe that the same result should hold also in the non-Galois case.

ii) We do not know any example of a possible (very) simple configuration with $m > 3(g-1)$.

Proof. First of all let us assume that $m = 3(g-1)$ and let us calculate $\mathfrak{a}(\mathcal{A}) - 8/3$ in this case.

$$\begin{aligned}\mathfrak{a}(\mathcal{A}) - 8/3 &= \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{1}{r_i^2}}{\frac{2}{3} + 1 - \frac{1}{m}\sum_{i=1}^{m} \frac{1}{r_i}} - \frac{2}{3} \\ &= \frac{3 - \frac{3}{m}\sum_{i=1}^{m} \frac{1}{r_i^2} - \frac{10}{3} + \frac{2}{m}\sum_{i=1}^{m} \frac{1}{r_i}}{5 - \frac{3}{m}\sum_{i=1}^{m} \frac{1}{r_i}}\end{aligned}$$

and if we denote by m_k the number of components D_i of D which have branching multiplicity $r_i = k$

$$= \frac{-\frac{1}{3} + \frac{1}{m}\sum_k \left(\frac{2m_k}{k} - \frac{3m_k}{k^2}\right)}{5 - \frac{3}{m}\sum_k \frac{m_k}{k}} = \frac{-\frac{1}{3} + \frac{1}{m}\sum_k m_k \frac{2k-3}{k^2}}{5 - \frac{3}{m}\sum_k \frac{m_k}{k}}$$

The expression $\frac{2k-3}{k^2}$ has a global maximum in $k = 3$ and which yields the inequality

$$-\frac{1}{3} + \frac{1}{m}\sum_k m_k \frac{2k-3}{k^2} \leq -\frac{1}{3} + \frac{1}{m}\sum_k m_k \frac{3}{9} = 0$$

for the numerator. Equalitiy holds if and only if $m_3 = m$ and all other m_k 's are zero.

Since the denominator is always positive we conclude $\mathfrak{a}(\mathcal{A}) - 8/3 \leq 0$ with equality if and only if $m_3 = m$ and all other m_k 's are zero.

It remains to show that the abstract slope can only decrease if $m < 3(g-1)$: this follows by induction from the next lemma. □

Lemma 5.8 *Let $\mathcal{A} = (D_1 \cup \cdots \cup D_{m+1}, d, \{t_i, r_i, n_i\})$ be a simple configuration, Galois and with $m \leq 4(g(B_2) - 1)$. Let $\mathcal{A}' = (D_1 \cup \cdots \cup D_m, d, \{t_i, r_i, n_i\})$ be the configuration obtained by omitting the last component. Then $\mathfrak{a}(\mathcal{A}') < \mathfrak{a}(\mathcal{A})$.*

Proof. Using again the formulae from proposition 3.1 we calculate

$$\mathfrak{a}(\mathcal{A}') - 2 < \mathfrak{a}(\mathcal{A}) - 2$$

$$\Leftrightarrow \left(m - \sum_{i=1}^{m} \frac{1}{r_i^2}\right)\left(2g - 2 + m + 1 - \sum_{i=1}^{m+1} \frac{1}{r_i}\right) < \left(m + 1 - \sum_{i=1}^{m+1} \frac{1}{r_i^2}\right)\left(2g - 2 + m - \sum_{i=1}^{m} \frac{1}{r_i}\right)$$

$$\Leftrightarrow \left(m - \sum_{i=1}^{m} \frac{1}{r_i^2}\right)\left(1 - \frac{1}{r_{m+1}}\right) < \left(1 - \frac{1}{r_{m+1}^2}\right)\left(2g - 2 + m - \sum_{i=1}^{m} \frac{1}{r_i}\right)$$

$$\Leftrightarrow \mathfrak{a}(\mathcal{A}') - 2 = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{1}{r_i^2}}{\frac{2g-2}{m} + 1 - \frac{1}{m}\sum_{i=1}^{m} \frac{1}{r_i}} < \frac{1 - \frac{1}{r_{m+1}^2}}{1 - \frac{1}{r_{m+1}}} = 1 + \frac{1}{r_{m+1}}$$

The denominator on the left is bigger or equal to one since $\frac{2g-2}{m} \geq \frac{1}{2}$ and $r_i \geq 2$. Hence the left hand side is smaller than one which is strictly smaller than the right hand side and we are done. □

Example 5.9 — We want now to construct an example of a double Kodaira fibration which actually realizes the slope $8/3$ thereby proving Theorem A. First of all we construct the curve mentioned in Remark 5.4.

Let P_1, P_2, P_3 be distinct points in $\mathbb{P}^1$ and let γ_1, γ_2, γ_3 be simple geometrical loops around these points. The fundamental group $\pi_1(\mathbb{P}^1 \setminus \{P_1, P_2, P_3\})$ is generated by the γ_i's with the relation $\gamma_1\gamma_2\gamma_3 = 1$.

Consider in $\mathbf{Sl}(2, \mathbb{Z}/3\mathbb{Z})$ the elements

$$g_1 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

and define $\rho : \pi_1(\mathbb{P}^1 \setminus \{P_1, P_2, P_3\}) \to \mathbf{Sl}(2, \mathbb{Z}/3\mathbb{Z})$ by $\gamma_i \mapsto g_i$. This map is well defined and surjective, because g_1 and g_2 generate $\mathbf{Sl}(2, \mathbb{Z}/3\mathbb{Z})$ and $g_1g_2g_3 = 1$.

We define B to be the ramified Galois cover of $\mathbb{P}^1$ associated to the kernel of ρ. By construction $\mathbf{Sl}(2, \mathbb{Z}/3\mathbb{Z})$ acts on B as the Galois group of the covering and by the Riemann-Hurwitz formula

$$g(B) = \frac{|\mathbf{Sl}(2, \mathbb{Z}/3\mathbb{Z})|}{2}\left(\sum_{i=1}^{3}\left(1 - \frac{1}{ord(g_i)}\right) - 2\right) + 1 = \frac{24}{2}\left(1 - \frac{1}{3} - \frac{1}{3} - \frac{1}{4}\right) + 1 = 2$$

The subset

$$\mathcal{S} = \left\{\phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \phi_2 = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \phi_3 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\right\} \subset \mathbf{Sl}(2, \mathbb{Z}/3\mathbb{Z})$$

satisfies the conditions of Lemma 5.2 since ϕ_2, ϕ_3 and $\phi_3 \circ \phi_2^{-1}$ have no fixed points being of order six. This gives us $3 = 3(g(B)-1)$ graphs of automorphisms in $B \times B$ which do not intersect and we denote the corresponding divisor by D.

In order to use the tautological construction we have to construct a ramified covering of a curve of genus two minus three points (which we denote for the sake of simplicity by $B \setminus D$) and Proposition 5.6 tells us that the branching indices should all be equal to three in order to obtain the maximal possible slope.

Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be generators for $\pi_1(B)$ and let γ_1, γ_2, γ_3 simple geometrical loops around the three points. Then

$$\pi_1(B \setminus D) =< \alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \gamma_2, \gamma_3 > /(\Pi[\alpha_i, \beta_i] = \gamma_1\gamma_2\gamma_3)$$

is a free group and we can define a map

$$\begin{aligned} \rho : \pi_1(B \setminus D) &\to \mathbb{Z}/3\mathbb{Z} \\ \gamma_i &\mapsto 1 \\ \alpha_i, \beta_i &\mapsto 0 \end{aligned}$$

which induces the desired ramified covering $F \to B$.

At this point we use the tautological construction, but we observe that in this case only the first étale covering $B' \to B$ is needed.

Indeed, the divisor $D = D_1 + D_2 + D_3$ has degree 3 on each fibre of the first projection $p : B \times B \to B$ and the homomorphism ρ determines a simple cyclic covering of the fixed fibre $B^0 := \{x_0\} \times B$, ramified on the divisor $D \cap B^0$.

Therefore there is a divisor M on $B \cong B^0$ such that the simple cyclic covering is obtained by taking the cubic root of D in the line bundle corresponding to M, and in particular the following linear equivalence holds:

$$3M \equiv D|_{B^0}.$$

This linear equivalence determines M up to 3-torsion, and the monodromy of M is the same as the monodromy of ρ.

Therefore, if we take as before the étale covering $B' \to B$ associated to the stabilizer of ρ, and denote by D' the pull back of D on $B' \times B$, then on $B' \times B$ the divisor $D' - 3p_2^*(M)$ is trivial on each fibre of the first projection p_1, hence there is a divisor L' on B' such that $D - 3p_2^*(M) = p_1^*(L')$.

By intersecting with the fibres of the second projection we find that $deg(L') = 0$, hence there is a divisor M' on B' such that $L' \equiv 3M'$, and we conclude that on $B' \times B$ we have the linear equivalence

$$D' \equiv 3(p_2^*(M) + p_1^*(L'))$$

and we can take the corresponding simple cyclic covering branched on D' inside the line bundle corresponding to the divisor $p_2^*(M) + p_1^*(L')$.

We obtain in this way a double étale Kodaira fibration which is in fact a Standard Kodaira fibration by Lemma 4.4. In particular we have a Kodaira fibration with base curve B' and with fibre of genus $g = 7$ (since $2g - 2 = 3 \cdot 2 + 3 \cdot 2$).

Since the associated ramified covering is branched exactly over D' with branching index three at each component, the formula for the slope of a simple configuration calculated in Proposition 3.1 yields

$$\frac{c_1^2(S)}{c_2(S)} = 2 + \frac{1 - \frac{1}{3}\sum_{i=1}^{3} \frac{1}{3^2}}{-\frac{e(B)}{3} + 1 - \frac{1}{3}\sum_{i=1}^{3} \frac{1}{3}} = \frac{8}{3}.$$

Remark 5.10 — We can also use this construction to give a partial answer to the question raised in 5.5. Knowing that the slope of a Kodaira fibration is strictly smaller than 3 it follows that $m < 8(g-1)$. In fact, via a suitable base change we obtain a divisor $D' \subset B' \times B$ such that

(*i*) if m is odd, then there is a component D_1 mapping to B' with degree one,
(*ii*) setting $D'' := D'$ if m is even, and $D'' := D' - D_1$ if m is odd, then
(*iii*) we can take a double cover branched over D''.

The Kodaira fibration constructed in this way turns out, under the assumption $m \geq 8(g-1)$, and in view of the above formulae, to have a slope ≥ 3: this is a contradiction.

It follows in particular as a consequence: if B is a curve of genus 2 and we have 8 étale maps from a fixed curve C of arbitrary genus to B, then two of them have a coincidence point.

6. The moduli space

This section is devoted to the description of the moduli space of double étale Kodaira fibrations. We start with some lemmas.

Lemma 6.1 *Let B_1, B_2 be curves of genus $b_i \geq 2$ resp. and let $C \subset B_1 \times B_2$ be an irreducible curve. Then*

- *C is smooth and the restricted projections $p_i : C \to B_i$ are étale if and only if*
- *the negative of the selfintersection of C attains its maximum possible value, i.e., if and only if*

$$-C^2 = 2m_i(b_i - 1) \qquad (i = 1, 2)$$

where $m_1 = C \cdot \{*\} \times B_2$ and $m_2 = C \cdot B_1 \times \{*\}$.

Proof. "$\Rightarrow$" We calculated this at the beginning of section 3.1.
"$\Leftarrow$" Let $p = p(C)$ be the arithmetic genus of C. Then

$$\begin{aligned} 2p - 2 =& K_{B_1 \times B_2} \cdot C + C^2 = 2(b_1 - 1)m_1 + 2(b_2 - 1)m_2 - 2(b_j - 1)m_j \\ =& 2m_i(b_i - 1) \qquad (i \neq j) \end{aligned}$$

Let $\tilde{C} \to C$ be the normalization and let $g = g(\tilde{C})$ be the geometric genus of C. We have $2p - 2 \geq 2g - 2$ by the normalization sequence and on the other hand $2g - 2 \geq 2m_i(b_i - 1) = 2p - 2$ by the Hurwitz formula for the projection $C \to B_i$. Hence $g = p$, C is smooth and equality holds in the last inequality, i.e., there is no ramification and the maps p_i are étale. □

Remark 6.2 — In general we see that $K_{B_1 \times B_2} \cdot C + C^2 = 2m_i(b_i - 1) + 2\delta + \rho_i$ where δ is the 'number of double points' and ρ_i is the total ramification index of $C \to B_i$. So

$$-C^2 = 2m_j(b_j - 1) - 2\delta - \rho_i \qquad (i \neq j)$$

Lemma 6.3 *Assume that we have a family of effective divisors $(D_t)_{t \in T}$, $D_t \subset (B_{1,t} \times B_{2,t})$, parametrized by a smooth curve T and such that the special fibre $D := D_0 = nC$ with C as in Lemma 6.1. If D' is another fibre ($D' = D_t$ for some t), then D' is of the same type $D' = nC'$ (the integer n being the same as before).*

Proof. Write $D' = \sum_j r_j C_j$ as a sum of irreducible components, so that $C_i \cdot C_j \geq 0$ for $i \neq j$. Write also $m_1^j = C_j \cdot (\{*\} \times B_{2,t})$ and $m_2^j = C_j \cdot (B_{1,t} \times \{*\})$.

We compare the self-intersection of the general fibre,

$$\begin{aligned} -{D'}^2 &= \sum_j r_j^2(-C_j^2) - 2\sum_{i \neq j} r_i r_j C_i \cdot C_j \\ &\leq \sum_j r_j^2(-C_j^2) \leq \sum_j r_j^2 2m_i^j(b_i - 1), \end{aligned}$$

and the central fibre (using the same notation as in Lemma 6.1),

$$-D'^2 = -D^2 = n^2(-C^2) = n^2 2m_i(b_i - 1),$$

obtaining

$$\sum_j r_j^2 m_i^j \geq n^2 m_i. \tag{1}$$

Since D and D' have the same intersection number with a horizontal (resp. vertical) curve we have

$$nm_i = \sum_j r_j m_i^j. \tag{2}$$

Every component of the general fibre C_i tends to a positive multiple of the curve C underlying the special fibre D and comparing again intersection numbers yields

$$m_i^j \geq m_i \tag{3}$$

Combining (1), (2) and (3) we get

$$\sum_j r_j^2 {m_i^j}^2 \geq n^2 m_i^2 = \left(\sum_j r_j m_i^j\right)^2 \geq \sum_j r_j^2 {m_i^j}^2.$$

Hence equality holds which, in the last step, is only possible if there is only one summand, i.e., $D' = n'C'$ for some irreducible curve C'.

The (in)equalities (1), (2) and (3) read now

$$n^2 m_i \leq n'^2 m_i', \qquad nm_i = n'm_i', \qquad m_i \leq m_i'.$$

Combining the two inequalities with the equality in the middle we get $n \leq n' \leq n$, hence $n = n'$.

Observing that C' fullfills the conditions of Lemma 6.1 we conclude the proof. □

Theorem 6.4 *Being a double étale Kodaira fibration is a closed and open condition in the moduli space.*

Proof. Since we know that the property of being a double Kodaira fibration is open and closed in the moduli space we can deduce from the previous Lemma that the condition of being double étale is actually open.

It remains to show that it is also closed for which it suffices to show that it is closed under holomorphic 1-parameter limits (see e.g. [Cat04], Lemma 2.8).

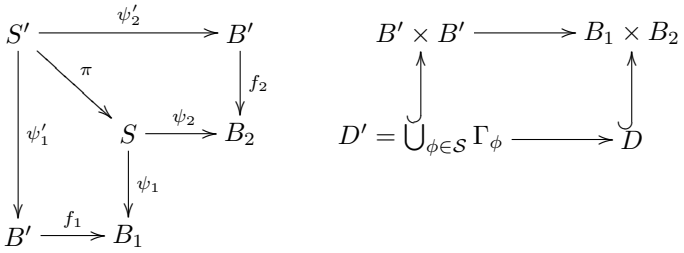
Assume that we have a 1-parameter family of surfaces with general fiber S_t a double étale Kodaira fibration. By the topological characterization (Proposition 2.5) also the special fibre S_0 is a double Kodaira fibration and we have to show that is in fact double étale.

By Lemma 2.8, we may assume that S_t is a branched covering of $B_{1,t} \times B_{2,t}$ branched over $D_t = \sum_i k_i D_{i,t}$, where the $D_{i,t}$'s are disjoint graphs of étale maps $\phi_i : B_{1,t} \to B_{2,t}$.

Now, $S_0 \to B_{1,0} \times B_{2,0}$ is branched over $D_0 := \sum_i k_i \nu_i D_{i,0}$ where $D_{i,t}$ tends to $\nu_i D_{i,0}$. Since however $D_{i,t}.(B_{1,t} \times \{*\}) = 1$ we have $\nu_i D_{i,0}.(B_{1,0} \times \{*\}) = 1$ which implies $\nu_i = 1$. Hence $D_{i,0}$ is the graph of a map $\phi_i' : B_{1,0} \to B_{2,0}$ and another application of Lemma 6.1 shows that also ϕ_i' is étale and S_0 is a double étale Kodaira fibration. □

We can now describe the moduli space of Standard Kodaira fibrations in detail. Let S be a Standard Kodaira fibration: then there exists a minimal common Galois

cover B' of B_1, B_2 yielding an étale pullback S' which is very simple. We call B' the *simplifying covering curve*. We have diagrams



where D is the ramification divisor of $\psi = \psi_1 \times \psi_2$ and $D' = \pi^* D$ is made of the graphs of a set of automorphisms $\mathcal{S} \subset \mathrm{Aut}(B')$. If we denote the Galois group of f_i by G_i $(i = 1, 2)$ the following holds:

Theorem 6.5 *Let S be a standard Kodaira fibred surface and let $\mathfrak{N}$ be the irreducible (and connected) component of the moduli space containing $[S]$. $\mathfrak{N}$ is then isomorphic to the moduli space of the pair (B', G), where B' is the simplifying covering curve defined above and G is the subgroup of $\mathrm{Aut}(B')$ generated by G_1, G_2 and $\mathcal{S}$.*

Proof. Let us first consider the case where $S = S'$, i.e., where S itself is very simple. By proposition 2.5 every deformation in the large of S' is a branched cover of a product surface $B_1 \times B_2$. Moreover, clearly $B_1 = B_2$ if

$(*)$ there is a component of the branch locus mapping to both curves B_1, B_2 with degree 1.

So let $(S_t)_{t \in T}$, be a family with connected parameter space T, having S' as a fibre. It is clear that the set of points of T where $(*)$ holds is open.

It is also closed because in the proof of theorem 6.4 we have seen that the type of the branch divisor remains the same under specialization and therefore the connected component $\mathfrak{N}$ parametrizes very simple Kodaira fibrations, i.e., branched coverings of a product $B \times B$ branched over the union of graphs of automorphisms.

The automorphisms defining the components of the branch divisors for different surfaces in $(S_t)_{t \in T}$ are clearly pairwise isotopic to each other and therefore we obtain a family of curves with automorphisms.

For each curve let G be the finite group generated by these automorphisms. This group has a faithful representation on the fundamental group of the curve, and therefore the group G remains actually constant.

G is a finite group and we have a faithful action on Teichmüller space $\mathfrak{T}_b$. We use now Lemma 4.12 of [Cat00] (page 29) to the effect that the fixed locus of this action is a connected submanifold (diffeomorphic to an Euclidean space), hence the moduli space of such pairs (B,G) is irreducible.

Viceversa any element in this moduli space gives rise to a complex structure on the differentiable manifold underlying S'.

Consider now the general case. It is clear that any deformation of S induces a deformation of $B' \to B_i$, hence any deformation of S yields a deformation of the pair (B', G).

Conversely, any deformation of the pair (B', G) yields a deformation of the pair $D' \subset B' \times B'$ such that the group $G_1 \times G_2$ leaves D' and the monodromy of the unramified covering of $(B' \times B') - D'$ invariant. □

Corollary 6.6 *There exist double étale Kodaira fibred surfaces which are rigid.*

Proof. Take the fibration constructed in Example 5.9: the automorphisms corresponding to the ramification divisor generate the whole triangle group of type

(3,3,4) and it is well known that pairs (B, G) yielding a triangle curve are rigid. Similarly for the other examples in proposition 5.3 which yield $m = 3(g-1)$. □

7. Rigid maps to the moduli stack of curves

In this section we want to interpret some of our results in terms of curves in the moduli space of curves.

One considers the moduli functor

$$\Psi : (Schemes) \to (Sets)$$
$$X \mapsto \{\text{flat families of smooth curves of genus } g \text{ over } X\}.$$

This functor is not representable in the category of schemes but there exists an algebraic stack $\mathcal{M}_g$ which is a fine moduli space of curves of genus g (see e.g. [HM]). In other word we have an isomorphism of functors from the category of schemes to the category of sets

$$\Psi(-) \cong \mathrm{Hom}(-, \mathcal{M}_g)$$

and there is a universal family $\mathcal{C}_g \to \mathcal{M}_g$ such that any flat family of smooth curves $Y \to X$ arises via a pullback diagram

$$\begin{array}{ccc} Y \cong \mu^*\mathcal{C}_g & \longrightarrow & \mathcal{C}_g \\ \downarrow & & \downarrow \\ X & \xrightarrow{\mu} & \mathcal{M}_g. \end{array}$$

We consider now the Kodaira fibration $S \to B'$ constructed in Example 5.9 which is a smooth fibrations of smooth curves of genus 7. By the universal property we have a corresponding moduli map

$$\mu : B' \to \mathcal{M}_7$$

which is not constant.

Proposition 7.1 *The map $B' \xrightarrow{\mu} D := \mu(B') \subset \mathcal{M}_7$ is rigid in the following sense: If*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\tilde{\mu}} & \mathcal{M}_7 \\ \downarrow & & \\ X & & \end{array}$$

is any connected family of smooth curves in $\mathcal{M}_7$ such that for some point $x_0 \in X$ the map $\tilde{\mu}_{x_0} : \mathcal{D}_{x_0} \to \mathcal{M}_7$ coincides with μ then $\tilde{\mu}_x = \mu$ for all $x \in X$ and the image of $\tilde{\mu}$ is equal to D.

Proof. This follows directly from Corollary 6.6: The map $\tilde{\mu}$ corresponds to a family of curves over $\mathcal{D}$ and, since $\mathcal{D} \to X$ is a family of curves, this yields a family of smooth surfaces

$$\mathcal{S} = \tilde{\mu}^*\mathcal{C}_7 \to X.$$

By our assumption the surface $\mathcal{S}_{x_0}$ over the point $x_0 \in X$ is isomorphic to S. Since S is rigid we have $\mathcal{S}_x \cong S$ for all $x \in X$ and moduli maps coincide. □

Acknowledgements

The present research took place in the realm of the D.F.G. Schwerpunkt " Globale Methoden in der komplexen Geometrie".

The first author would like to thank Joe Harris for an interesting conversation, Janos Kollár for remark 2.9, and Gianpietro Pirola for remark 2.6.

The second author wishes to thank all the members of the Lehrstuhl Mathematik II at the University of Bayreuth, especially Ralf Gugisch, for useful discussions concerning the computational problems.

Finally, the referee observes an interesting coincidence between the fact that our Kodaira fibrations with the highest slope $\nu = 8/3$ are rigid, and some results of Igor Reider ([Rei05]). Reider shows that, if the slope of a surface S satisfies $\nu \geq 8/3$, then not only the surface S has 'few moduli', but also one can analyse the structure of the local moduli space. The investigation of this interesting analogy seems however not easy.

References

[At69] M. F. Atiyah, *The signature of fibre-bundles.* Global Analysis (Papers in Honor of K. Kodaira) Univ. Tokyo Press, Tokyo, (1969), 73–84.

[BHPV] W. Barth, K. Hulek, C. Peters, A. van de Ven, *Compact Complex Surfaces*, 2nd edition, Springer-Verlag, 2004.

[B-D02] J. Bryan, R. Donagi, *Surface bundles over surfaces of small genus.* , Geom. Topol. 6 (2002), 59–67

[Cat00] F. Catanese, *Fibred surfaces, varieties isogenous to products of curves and related moduli spaces*, Amer. J. Math. 122 (2000), 1-44.

[Cat03] F. Catanese, *Fibred Kähler and quasi projective groups*, Adv. Geom. (2003), suppl., S13–S27.

[Cat04] F. Catanese, *Deformation in the large of some complex manifolds I*, Ann. Mat. Pura Appl. (4), 183(3):261–289, 2004.

[Cat06] F. Catanese, *Surface classification and local and global fundamental groups. I.*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. , Rend. Lincei (9) Mat. Appl. 17 (2006), no. 2, 135–153.

[CHS57] S.S. Chern, F. Hirzebruch, J.P. Serre, *On the index of a fibred manifold*, Proc. Amer. Math. Soc. 8 (1957), 587-596.

[F-G65] W. Fischer, H. Grauert, *Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, (1965), 89–94.

[GD-H91] G. González Díez,W. J. Harvey, *On complete curves in moduli space. I, II.* , Math. Proc. Cambridge Philos. Soc. 110 (1991), no. 3, 461–466, 467–472.

[HM] J. Harris, I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics, 187. Springer-Verlag, New York, 1998.

[Hirz69] F. Hirzebruch, F. *The signature of ramified coverings.* Global Analysis (Papers in Honor of K. Kodaira) Univ. Tokyo Press, Tokyo (1969), 253–265.

[Iz03] T. Izawa, *Note on the Riemann-Hurwitz type formula for multiplicative sequences*, Proc. Amer. Math. Soc. 131 (2003), no. 11, 3583–3588.

[J-Y83] J. Jost; S.T. Yau, *Harmonic mappings and Kähler manifolds*, Math. Ann. 262 (1983), no. 2, 145–166.

[Kod67] K. Kodaira, *A certain type of irregular algebraic surfaces.* J. Analyse Math. 19 (1967), 207–215.

[Kot99] D. Kotschick, *On regularly fibered complex surfaces.* Proceedings of the Kirbyfest (Berkeley, CA, 1998), 291–298 (electronic), Geom. Topol. Monogr., 2, Geom. Topol. Publ., Coventry, 1999.

[Ks68] A. Kas, *On deformations of a certain type of irregular algebraic surface* Amer. J. Math. 90 (1968), 789–804.

[Liu96] K. Liu, *Geometric height inequalities*, Math. Res. Lett. 3 (1996), no. 5, 693–702.

[GAP04] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4; 2005. (http://www.gap-system.org)

[Rei05] I. Reider,, *Geography and the number of moduli of surfaces of general type.* Asian J. Math. 9 (2005), no. 3, 407–448.

[Zaal95] C. Zaal, *Explicit complete curves in the moduli space of curves of genus three*, Geom. Dedicata 56 (1995), no. 2, 185–196.

Author's address:

Dr. Sönke Rollenske
Lehrstuhl Mathematik VIII
Mathematisches Institut
Universität Bayreuth, NWII
D-95440 Bayreuth, Germany
e-mail: Soenke.Rollenske@uni-bayreuth.de

Prof. Dr. Fabrizio Catanese
Lehrstuhl Mathematik VIII
Mathematisches Institut
Universität Bayreuth, NWII
D-95440 Bayreuth, Germany
e-mail: Fabrizio.Catanese@uni-bayreuth.de